Communication in Mathematics

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These are the lecture notes for the course Communication in Mathematics given in the academic years 2018–2020. Please let me know by email at merry@math.ethz.ch if you spot any typos!

Contents

LECTURE 1

The Basics: Numbers, Symbols, Words

In this lecture we look at how numbers and symbols are used in English sentences to make readable, sensible, and correct mathematical statements. We identify common issues and offer ways to fix them. This is the first step towards writing well. The second step is forming a good writing habit. The problem sheet is designed to give you some guided practice, but we strongly encourage you to immediately start applying what you have learned to your daily maths work.

Some of the issues we discuss may seem obvious, but before disregarding them, ask yourself whether your work ever contains them. A quick glance at any recent formal piece of writing should give you an objective answer.

In Section 1.1 we work through four examples that illustrate the issues we want to focus on. In Section 1.2 we offer a list of the issues we have covered. At the end of the lecture there is a Problem Sheet.

1.1 Worked Examples

In this section we work through four examples. The issues they illustrate will be summarised in Section 1.2, and you may wish to read that section first. However, it is imperative that you read through the worked examples too, because they offer a more detailed discussion, a wider perspective, and they show the process you would go through to resolve the issues.

EXAMPLE 1.1. Consider the sentence: Three is the smallest odd prime.

i) State the issue with this sentence.

ii) Which of the four options below is a direct improvement? There is only one correct answer.

   A. 3 is the smallest odd prime.
   B. The smallest odd prime is three.
   C. The smallest odd prime is 3.
   D. There is only one smallest odd prime and it is 3.

iii) Explain why the other three answers are not improvements.

SOLUTION. Read through carefully. The solution contains other examples and is instructive.

i) The issue: The number 3 is written as a word. This may not cause confusion in our example, but it easily causes confusion elsewhere, especially when the number is small e.g. 0,1,2, or 3. For example, can you decipher the following sentence?

- BAD: There are 2 solutions of this equation and 1 of them is 3.

If you are using a number for counting, then you should write it out in full, unless it's a large number (e.g. There are 153 solutions). If you are referring to a specific number, then you should use numerals.

- GOOD: There are two solutions of this equation and one of them is 3.

It would be better still if you could rephrase the sentence to make it more elegant, though this requires additional information.

- BETTER: The equation $x^2 - 9 = 0$ has two solutions: $x = \pm 3$.

ii) Answer C is correct.

iii) Let's consider the other answers *one by one*. (Note that we must not write: *1 by 1!* That should be reserved for talking about unit squares.)

- A. Here the number has correctly been written as a numeral. However, in English we try not to start a sentence with a numeral if at all possible.
- B. Here the number has been written out in words, whereas it should been a numeral as in C.
- D. Here the issue is subtler: we are faced with redundancy. In fact, this an example of two different types of redundancy, which you would do well to distinguish.
  - Mathematical redundancy: The smallest odd prime is a well-defined, unique number. Therefore, the words *only one* are not needed.
  - Linguistic redundancy: The words *and it is* are unnecessary as is shown by the elegant phrasing offered in C.

Mathematical redundancy is usually a sign that you do not understand the underlying mathematics properly, and have therefore added unnecessary conditions or "padding" to your statements. A degree of linguistic redundancy is expected in the following situations, ordered from most to least: free speech (impromptu conversations), informal verbal expositions (student seminars), formal verbal expositions (professional seminars), informal work or work under time pressure (homework, exams). Very little redundancy is expected in formal written work (theses, papers). But don't worry, it is all a matter of habit!

Example 1.2. Improve the following sentence:

$$\text{Let } f(x) \colon = x^2 - 9,\ f(x) = 0 \Rightarrow x \text{ is either 3 or } -3$$

Solution. There are many ways to improve exposition. Here the process is broken down into steps that identify specific issues before fixing them. You may find it helpful to work through examples using a similar process yourself. Eventually this will become second nature and you will either write this way by default or correct any issues in your writing upon revising without having to think about it.

i) All formal written mathematics must read like correct English. This includes sentences with symbols. Therefore, a basic requirement is proper punctuation that will help the reader understand the structure of a statement. In our sentence the segment

$$\text{Let } f(x) := x^2 - 9$$

is a complete sentence, and therefore should end in a period. Likewise, our sentence ends after -3, and therefore we should put a period. After implementing those changes we have the following:

$$\text{Let } f(x)\colon = x^2 - 9.\ f(x) = 0 \Rightarrow x \text{ is either } 3 \text{ or } -3.$$

ii) Now the next issue becomes apparent; this jumble of symbols

$$x^2 - 9.\ f(x) = 0$$

is confusing. As we saw in the previous example, in English you do not want to start a sentence with a symbol if you can help it. Similarly, any adjacent formulas should be separated by words. So what should we add? The implication operator $\Rightarrow$ offers a clue. Whenever we have an implication in mathematics, we can make a sentence of the form *if something, then something else.* If we rephrase our sentence using this form, we get the following:

$$\text{Let } f(x)\colon = x^2 - 9. \text{ If } f(x) = 0, \text{ then } x \text{ is either } 3 \text{ or } -3.$$

Note that in mathematics the comma in the form *if..., then...* is required by convention.

iii) If we stopped editing at the previous stage, the sentence would be understandable, but clunky. The function $f(x)$ does not need to be defined. Indeed, in general, unless a symbol is necessary, you should not introduce it into your writing. Let us remove the definition of f and replace f in the second sentence.

$$\text{If } x^2 - 9 = 0, \text{ then } x \text{ is either } 3 \text{ or } -3.$$

iv) We have reduced the monstrosity from the beginning to something that a reader can easily parse. It is a matter of finesse now. The statement *if...,* *then...* is usually used for branching into different options. For example you would say:

If the function has a single positive root, then...

followed by

If it does not, then...

But this is not what we are aiming to achieve with our sentence. We are making a statement about a concrete equation; there is nothing conditional about it. Therefore one way to resolve this would be to rephrase it as follows:

$$\text{The equation } x^2 - 9 = 0 \text{ has two solutions: } x = \pm 3.$$

You may recognise this as the "BETTER" solution from above.

The following two examples focus on redundancy. As we will see throughout the course, a key element of writing well is knowing how to balance notation and words. Your exposition can be mathematically correct but if it is laden with too many words or with too many symbols the reader will struggle to understand the content.

EXAMPLE 1.3. Rewrite the following statement to remove the symbols.

> Let Y be a compact metric subspace of a space Z. A continuous, $\mathbb{R}$-valued function f on Y has both a minimum and a maximum value.

SOLUTION. Take a moment to understand the statement, then work through it step by step.

- The second sentence uses only the definition of Y. The definition of Z is unnecessary. Remove it.

  > Let Y be a compact metric space. A continuous, $\mathbb{R}$-valued function f on Y has both a minimum and a maximum value.

- The mixed expression $\mathbb{R}$-*valued* can be written in words and it is useful to do so in most cases.

  > Let Y be a compact metric space. A continuous, real-valued function f on Y has both a minimum and a maximum value.

- The function is labelled as f, but the label is not used elsewhere. Drop it.

  > Let Y be a compact metric space. A continuous, real-valued function on Y has both a minimum and a maximum value.

- The space Y is defined succinctly and used immediately. Combine the two sentences to remove the symbol.

  > A continuous, real-valued function on a compact metric space has both a minimum and a maximum value.

As it turns out none of the symbols in the original statement were necessary. This is often the case with general results or basic theorems.

EXAMPLE 1.4. Rewrite the following statement to make it as concise as possible.

> We then conclude that the set B will have no element of the set A contained in it.

SOLUTION. Aside from being wordy, this statement is written in the future tense. Mathematical statements should be universal truths, and are traditionally stated in the present tense.

- Remove the future tense.

  > We then conclude that the set A <u>*has*</u> no element of the set B contained in it.

- Whenever you can write one word instead of many, do so. Specifically, *We then conclude that* can be replaced with *Therefore*.

  Therefore the set A has no element of the set B contained in it.

- Look out for ways to simplify the language. What you write down first will often not be the most straightforward phrasing. Here, *has no element contained* should be *contains no element*.

  Therefore A *contains no element* of the set B.

- Think about what the statement is saying. How would a mathematician talk about elements in the intersection of two sets?

  Therefore the sets A and B *are disjoint*.

- Our reduced statement is elegant and clear, and in an expository setting it may be best to leave it as it stands. However, in a proof for example, you might want to reduce it further.

  Therefore $A \cap B = \emptyset$.

1.2 Guidelines

In this section we list some of the common issues that occur when writing about numbers, symbols, and formulas. We are primarily concerned with *inline notation*, that is, notation that appears on the same line as non-specialised words and that forms part of an ordinary sentence.

In contrast, *displayed equations* are equations that are written in a separate line and contain few, if any, words. Displayed equations are easier to write without special training; you see them all the time in lectures and they are the mathematical shorthand you use to manipulate symbols. At first it may not seem obvious which formula to display and when (size is a good guideline: the larger the more likely it needs its own line!), but it is something you will pick up with practice.

1.2.1 Proper punctuation

Every sentence should read as correct English. If you are struggling to determine where the punctuation should go, read the sentence aloud, including the numbers and symbols. Periods and commas are fundamental to the parsing of written language, so prioritise getting these right. In future lectures we will address punctuation in general; for now make sure you understand the differences between these examples.

- BAD: Let $f(x) := x^2$ and $g(x) := x^3 - 1$
- BAD: Let $f(x) := x^2$, and $g(x) := x^3 - 1$.
- BAD: let $f(x) := x^2$ and $g(x) := x^3 - 1$.
- GOOD: Let $f(x) := x^2$ and $g(x) := x^3 - 1$.

Displayed equations also have to read as properly punctuated sentences. However in displayed equations commas, colons, and spaces become syntactic elements, that is, they can stand in for certain words. Inline equations use words to make the sentence flow between notation:

- GOOD: For all $n > 2$, the sequences coincide: $s_n = t_n$.
- BAD: We have $\forall n > 2$, $s_n = t_n$.

Displayed equations are compressed:

- BAD: We have:

$$\text{for all } n > 2 \text{, the sequences coincide: } s_n = t_n.$$

- GOOD: We have:

$$\forall n > 2, \quad s_n = t_n.$$

- BETTER: We have:[1]

$$s_n = t_n, \quad \forall n > 2.$$

1.2.2 Adjacent formulas

Adjacent formulas should be separated with words.

- BAD: Let $f(x) := x^2\ g(x) := x^3 - 1$.
- BAD: Let $f(x) := x^2,\ g(x) := x^3 - 1$.
- GOOD: Let $f(x) := x^2$ and $g(x) := x^3 - 1$.

1.2.3 Beginning sentences with words

In English we do not begin sentences with symbols or numbers.

- BAD: g is a continuous function.
- GOOD: The function g is continuous.

1.2.4 Operators and words

Operators should not be combined with words in a sentence.

- BAD: If $x > 0$, then $f(x)$ is $\leq$ zero.
- BAD: x is positive $\Rightarrow$ we have $f(x)$ is $\leq$ zero.
- GOOD: If $x > 0$, then $f(x) \leq 0$.

[1] This is a more standard ordering. Often you cannot simply convert words into symbols or vice versa without changing the word order, if not also the sentence structure.

1.2.5 Unnecessary notation

Omit all symbols that are not strictly needed. To do this, you must be able to identify them. Firstly, check whether there are any symbols that used just once. Those can be immediately deleted.

- BAD: Every smooth function f is continuous.
- GOOD: A smooth function is continuous.

Secondly, as in Example 1.2, check that you are not defining a symbol only to use it once or twice. In those cases, you may be able find a more elegant phrasing.

1.2.6 Necessary notation

As we saw above, unnecessary notation leaves the reader puzzling over symbols and trying to work out what is going on. In contrast, not introducing the necessary notation leaves the reader puzzling over cumbersome statements.

- BAD: Let us consider the function $\mathbb{R}$ to $\mathbb{R}$ such that for all values less than $-\frac{1}{2}$ it is equal 0, for all values in $[-\frac{1}{2}, 0]$ it is the identity, and for all positive values it is equal to 1.
- GOOD: Define $f\colon \mathbb{R} \to \mathbb{R}$ be as follows:

$$f(x) := \begin{cases} 0, & x < -\frac{1}{2}, \\ x, & x \in [-\frac{1}{2}, 0], \\ 1, & x > 0. \end{cases}$$

If you are intending to discuss a specific mathematical object, name it and define it as early as possible. In particular, you should not *first* discuss a function *then* define it.

- BAD: The function f is continuous but not continuously differentiable. Let $f(x) := |x|$.

Reverse the ordering of the two sentences. As a rule, define first, discuss next.

1.2.7 Small cardinal numbers

In linguistics, *cardinal numbers* count objects (zero, one, two, three, four apples). In mathematical writing, small (cardinal) numbers used counting for must be written out as words to distinguish from specific numbers that are always written as numerals.

- BAD: Is twenty-seven a prime number?
- GOOD: Is 27 a prime number?
- BAD: We split the proof into 3 parts.
- GOOD: We split the proof into three parts.

Note that large numbers can stay as numerals:

- BAD: There are five-hundred and three elements.
- GOOD: There are 503 elements.

For how to write ordinal numbers, see Section 1.2.9.

1.2.8 Special cases of *one*

The word *one* may appear as the gender-neutral, indefinite pronoun in sentences such as:

- One may assume...
- In that case, one can prove that...

In a future lecture we will discuss which pronoun (*I, you, we, one*) you are expected to use in a certain piece of written work, but for now just note that here one should definitely not write the number 1 when one means the word *one*.

Further, note that *one* may appear in other English phrases such as:

- one by one (as above in Example 1.2),
- to take one at a time,
- on the one hand.

There should not be much doubt about whether to write the numeral or the word in these cases. However, when these phrases are used in proximity to actual counting numbers or numerals the result can be confusing, and you must find a way to rephrase the sentence.

- BAD: One can show that here is one solution.
- BAD: We have considered the cases one by one and shown that are two positive solutions and one negative one.

1.2.9 Small ordinal numbers

In linguistics, *ordinal numbers* denote the location of an object within a list (zeroth, first, second, third, fourth, etc). Given a sequence relative to n, this is how you would write out its elements:

$$\ldots (n-1)\text{st}, n\text{th}, (n+1)\text{st}, (n+2)\text{nd}, (n+3)\text{rd}, (n+4)\text{th}, \ldots$$

The rule is that you append to $(n \pm i)$ the ending of the ordinal number i.

However, note that we also commonly use the ordinal numbers to indicate which part of a proof or solution we are referring to.

- In the first section, we show that...
- The second solution to this equation is...

Therefore, you should take some care when writing about lists. This also applies to the words *last* and *final*.

- BAD: In the first section, we show that the first element of the sequence is...
- BAD: The second solution is found in the second part of the proof.
- BAD: The last equation, fives us the last term of...

1.2.10 Firstly, secondly, thirdly

Complicated arguments are split into segments. Sometimes these segments are lemmas and propositions, but at other times, they can be as small as a few lines within a paragraph. It is crucial for the reader to be able to follow along, so certain keywords such as *first/firstly, secondly, thirdly, lastly* are used to indicate the start of a new segment. In a future lecture we will discuss the formal structures used to form mathematical statements, but for now just note that you should be careful when using these particular keywords in the vicinity of numbers.

- BAD: First, in the first section, we show that the first element is 1.
- BAD: First, in section I, we show that the 1st element is 1.[2]
- BAD: Secondly, the second solution of this equation is...
- BAD: Lastly, the last equation, gives us the last term of...

You get the idea.

[2] Writing *first* in three different ways is *not* a solution. It indicates that the writer noticed the problem, but chose to apply a lazy "fix". Bad writing can cause hours of headache, so spare a few minutes to improve it.

Problem Sheet 1

Turn off your math-brain! This problem sheet only tests the writing guidelines we have covered. The maths should not be wrong, but this is not the place to look for missing conditions or exceptions.

Problem 1. Which statements are well-written?

A. There are zero solutions to this equation.

B. There are 0 solutions to this equation.

C. Zero is a solution if this equation.

D. 0 is a solution of this equation.

E. There are more than ten to the seven solutions.

F. There are more than 10^7 solutions.

G. The lth and the $(l+1)st$ elements are the same.

H. The lth and the $(l+1)$st elements are the same.

I. The unit circle is given by the equation $x^2 + y^2 = 1$. It is called a unit circle because it has radius one.

J. The unit square is a one by one square.

K. Let us consider the cases one by one.

L. One first solves the first equation.

M. Suppose $A \cap B_i = \emptyset$, $\forall i$.

N. We will suppose that $A \cap B_i = \emptyset$ for all i.

Problem 2. Consider the following sentence:

The number square root of seven cannot be represented as a fraction.

i) State the issues with this sentence.

ii) Which of the options below is a direct improvement? There is only one correct answer.

  A. The square root of 7 is not equal to $\frac{a}{b}$ for any $a, b \in \mathbb{Z}$.

  B. $\sqrt{7}$ is not a fraction.

  C. The number $\sqrt{7}$ is irrational.

  D. The square root of seven is not a rational number.

E. $\sqrt{7}$ is irrational.

iii) Explain why the other three answers are not improvements.

For the following two problems complete steps i), ii), and iii) as above, using the given sentence and the suggested options.

Problem 3. $\sin\colon \mathbb{R} \to [-1,1]$ is surjective

A. $\sin\colon \mathbb{R} \to [-1,1]$ is surjective.

B. $\sin x\colon \mathbb{R} \to [-1,1]$ is surjective.

C. The function $\sin\colon \mathbb{R} \to [-1,1]$ is a surjective function.

D. For every $y \in [-1,1]$ there is an $x \in \mathbb{R}$ such that $\sin(x)\colon \mathbb{R} \to [-1,1]$ is surjective.

E. The sine function is surjective.

F. The sin function is surjective.

Problem 4. The circle $x^2 + y^2 = 1$ does not $\cap \{x = 5\}$.

A. The circle $x^2 + y^2 = 1$ does not $\cap$ the line defined by $x = 5$.

B. The unit circle does not intersect line which passes through $(5,0)$ and $(5,1)$.

C. $x^2 + y^2 = 1$ does not intersect the line $x = 5$.

D. The unit circle does not intersect the line $x = 5$.

E. The circle $x^2 + y^2 = 1$ does not intersect $l(x,y) := (5,y)$.

LECTURE 2

Patterns of Mathematical Statements

Language in general is rich and versatile—think of novels and poetry and jargon. Mathematical English is anything but. It is not your task to become the next Oscar Wilde. This is an example of what you should *not* be aiming for:

> Words! Mere words! How terrible they were! How clear, and vivid, and cruel! One could not escape from them. And yet what a subtle magic there was in them! They seemed to be able to give a plastic form to formless things, and to have a music of their own as sweet as that of viol or of lute. Mere words! Was there anything so real as words?
>
> (Excerpt from *The Picture of Dorian Gray*)

The task of written mathematics is to be clear and precise in conveying the underlying ideas. To achieve this, mathematical English has developed a range of conventional "phrase-patterns" used when speaking and writing. This is not to say that mathematicians are lazy or incapable of speaking properly, but rather that they choose to use most of their creative powers for proving theorems and not for performing word-acrobatics.

In this lecture, we discuss the common phrase-patterns used in mathematical exposition. In Section 2.1, we discuss two examples. In Section 2.2, we list the phrase-patterns according to their purpose. At the end of the lecture there is a Problem Sheet.

2.1 Examples

EXAMPLE 2.1. Consider the following theorem from the classical textbook *Topology* by James R. Munkres. Highlight the phrase-patterns characteristic to mathematical statements.

> **Theorem 6.2** *Let A be a set; suppose that there exists a bijection $f\colon A \to \{1,\dots,n\}$ for some $n \in \mathbb{Z}_+$. Let B be a proper subset of A. Then there exists no bijection $g\colon B \to \{1,\dots,n\}$; but (provided $B \neq \emptyset$) there does exist a bijection $h\colon B \to \{1,\dots,m\}$ for some $m < n$.*

SOLUTION. We go through them according to type.

1. Defining.

   > **Theorem 6.2** *Let A be a set; suppose that there exists a bijection $f\colon A \to \{1,\dots,n\}$ for some $n \in \mathbb{Z}_+$. Let B be a proper subset of A. Then there exists no bijection $g\colon B \to \{1,\dots,n\}$; but (provided $B \neq \emptyset$) there does exist a bijection $h\colon B \to \{1,\dots,m\}$ for some $m < n$.*

2. Making an assumption.

   Theorem 6.2 *Let A be a set; suppose that there exists a bijection $f\colon A \to \{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$. Let B be a proper subset of A. Then there exists no bijection $g\colon B \to \{1, \ldots, n\}$; but (provided $B \neq \emptyset$) there does exist a bijection $h\colon B \to \{1, \ldots, m\}$ for some $m < n$.*

3. Concluding.

   Theorem 6.2 *Let A be a set; suppose that there exists a bijection $f\colon A \to \{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$. Let B be a proper subset of A. Then there exists no bijection $g\colon B \to \{1, \ldots, n\}$; but (provided $B \neq \emptyset$) there does exist a bijection $h\colon B \to \{1, \ldots, m\}$ for some $m < n$.*

4. Contrasting.

   Theorem 6.2 *Let A be a set; suppose that there exists a bijection $f\colon A \to \{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$. Let B be a proper subset of A. Then there exists no bijection $g\colon B \to \{1, \ldots, n\}$; but (provided $B \neq \emptyset$) there does exist a bijection $h\colon B \to \{1, \ldots, m\}$ for some $m < n$.*

5. Restricting.

   Theorem 6.2 *Let A be a set; suppose that there exists a bijection $f\colon A \to \{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$. Let B be a proper subset of A. Then there exists no bijection $g\colon B \to \{1, \ldots, n\}$; but (provided $B \neq \emptyset$) there does exist a bijection $h\colon B \to \{1, \ldots, m\}$ for some $m < n$.*

EXAMPLE 2.2. Consider now the first part of the proof of the same theorem. Highlight the phrase-patterns characteristic to mathematical statements.

> *Proof.* The case in which $B = \emptyset$ is trivial, for there cannot exist a bijection of the empty set B with the nonempty set $\{1, \ldots, n\}$.
>
> We prove the theorem "by induction." Let C be the subset of $\mathbb{Z}_+$ consisting of those integers n for which the theorem holds. We shall show that C is inductive. From this we conclude that $C = \mathbb{Z}_+$, so the theorem is true for all positive integers n.

SOLUTION. Again we go through them according to type.

1. Branching and cases.

   Proof. The case in which $B = \emptyset$ is trivial, for there cannot exist a bijection of the empty set B with the nonempty set $\{1, \ldots, n\}$.

   We prove the theorem "by induction." Let C be the subset of $\mathbb{Z}_+$ consisting of those integers n for which the theorem holds. We shall show that C is inductive. From this we conclude that $C = \mathbb{Z}_+$, so the theorem is true for all positive integers n.

2. Defining.

Proof. The case in which $B = \emptyset$ is trivial, for there cannot exist a bijection of the empty set B with the nonempty set $\{1, \ldots, n\}$.

We prove the theorem "by induction." Let C be the subset of $\mathbb{Z}_+$ consisting of those integers n for which the theorem holds. We shall show that C is inductive. From this we conclude that $C = \mathbb{Z}_+$, so the theorem is true for all positive integers n.

3. Reasoning or explaining what will be shown.

Proof. The case in which $B = \emptyset$ is trivial, for there cannot exist a bijection of the empty set B with the nonempty set $\{1, \ldots, n\}$.

We prove the theorem "by induction." Let C be the subset of $\mathbb{Z}_+$ consisting of those integers n for which the theorem holds. We shall show that C is inductive. From this we conclude that $C = \mathbb{Z}_+$, so the theorem is true for all positive integers n.

Already these few sentences exhibit a few different ways to write down your reasoning. The Guidelines below give a much wider variety of phrase-patterns, but let us go through the most frequent ones here.

- *We prove/show...* is a fundamental way to state your intentions. In lectures you will often hear professors saying:
  - Today we (shall) prove...
  - Now let us prove...

  In written mathematics, as is illustrated by the example, it is difficult to go through a proof without using a variation of those words. Whether to indicate the type of proof (by induction, by example, etc) or to indicate a step along the way (we first prove the lemma).
  - We prove that f is continuous by applying...
  - Now we can prove that the statement holds in the following cases...
  - Let us develop the tools that we need to prove...

  The words *show* and *prove* can often be used as a synonym. So this would be fine:

  We shall prove that C is inductive.

  However, direct word substitution does not always work.
  - GOOD: We prove the lemma.
  - BAD: We show the lemma.

  Lastly, the following would not be wrong; it would just be clumsy. (Note that the word *proof* sneaked in there, hinting that perhaps the verb *to prove* would have a been better choice.)

  We show the theorem is true using a proof "by induction."

- *We conclude…* signals a conclusion drawn from an argument, like in the example. Though it sounds pretentious to conclude a statement holds if nothing has been shown.
  - GOOD: Applying the same argument we used in the proof of the Lemma, we conclude that X is a metric space with the desired properties.
  - BAD: If f is a smooth function, we conclude that f is continuous.

  In the last two examples, *then* would have been more appropriate.

  - GOOD: If f is a smooth function, then f is continuous.

  Between *we conclude…*, which indicates more important arguments, and *then…*, which indicates an immediate conclusion, there is a middle option: *it follows.*

- *It follows that…* is used to draw a direct conclusion from a preceding statement, more formally than *then…*. Like the others, but perhaps most obviously so, the phrase is taken to mean the implies symbol $\Rightarrow$.

  However, this phrase has a quirk: what does the *it* "point" towards? In general, *it* is a pronoun (like *he, she, we*) that stands in for a noun, so we do not have to repeat the noun, allowing us to make sentences such as this one:

  The <u>*function*</u> is smooth. Therefore <u>*it*</u> is continuous.

  Here *it* refers to *function*. But what about in this example?

  - From the preceding theorem, <u>*it*</u> follows that…

  Is *it* referring to the *preceding theorem*? Or to something else in the second example? The answer is that in these phrases *it* is a *dummy* pronoun that refers to nothing, but is part of the fixed phrase. This also happens in constructions using the passive voice:

  - *It was shown…*

  This example highlights another difference between phrases containing a dummy *it* and those containing *we*. Who is proving, concluding, saying that something follows from a preceding argument? You, the author are doing all those things, or perhaps the author of some other paper. By saying *we have shown* or *they have shown*, you draw attention to yourself or to those other authors. By saying *it was shown*, you step into the background and let the reader's attention be drawn to the mathematics. It may seem as if the latter were always preferred—for you are writing a proof or theorem, not a biography!—but this is not the case. Which pronouns to use will be covered in more detail in a future lecture, but here are a few examples where using this impersonal form of writing is not as helpful.

  You do not want to be caught mixing up your pronouns. (This is similar to not using *one* in a phrase and as a number; see Section 1.2.8 in

Lecture 1.) If in doubt, repeat the name of the object for which your pronoun stands.

- BAD: The function f is smooth. Then it follows that it is continuous.
- OK: The function f is smooth. Then it follows that f is continuous.

Whenever you can, however, try to reduce the number of words and symbols. This includes substituting a wordy expression with a shorter one, if appropriate.

- GOOD: The function f is smooth. It follows that f is continuous.
- BETTER: The function f is smooth, so it is continuous.

Finally, sometimes using a phrase just does not sound good. For example, can you see why the following statement is inferior to the one Munkres used?

> From this it follows that $C = \mathbb{Z}_+$, so the theorem is true for all positive integers n.

The first five words sound frivolous, like we are dealing with *this* and *that* and some unspecified *it* in the middle.

- *Because, as, since, for...* are (subordinating) conjunctions that introduce the reason that leads to a certain conclusion.
  - *Because* emphasises the reason. (e.g. Because my alarm didn't go off, I was late for class).
  - *As/since* assume the reason is well-understood (e.g. I can't be expected to understands this, as I'm only twenty years old).
  - *For* introduces the reason as an aside. You could almost put it in parentheses (e.g. I love mathematics, for it is beautiful).

  Consider the example from Munkres rewritten using the other options.

  - The case in which $B = \emptyset$ is trivial, because there cannot exist a bijection of the empty set B with the nonempty set $\{1, \ldots, n\}$.
  - The case in which $B = \emptyset$ is trivial, as/since there cannot exist a bijection of the empty set B with the nonempty set $\{1, \ldots, n\}$.
  - (ORIGINAL) The case in which $B = \emptyset$ is trivial, for there cannot exist a bijection of the empty set B with the nonempty set $\{1, \ldots, n\}$.

  They all work; the first is more emphatic and the final two are indistinguishable.

  For those who care, a grammar interlude: *Coordinating* conjunctions *and, or, but* are the words that allow you to build *compound* sentences with multiple *independent* clauses such as: G is a group and X_G is a set associated to that group. *Subordinating* conjunctions *because, as, when etc* allow you to build *complex* sentences with multiple *dependent* clauses such as: Because G is a group, we can associate to it a set X_G. Longer dependent clauses are usually set off by commas.

- *Accordingly, consequently, therefore, hence, thus, so . . .* are adverbs that introduce the conclusion (rather than the reason for the conclusion). The longer the adverb the more attention it will draw to itself, and the more weight it will give to the conclusion you are drawing. Let us rewrite the final sentence from Munkres's example using the options.
  - From this we conclude that $C = \mathbb{Z}_+$. Accordingly, the theorem is true for all positive integers n.
  - From this we conclude that $C = \mathbb{Z}_+$. Consequently, the theorem is true for all positive integers n.
  - From this we conclude that $C = \mathbb{Z}_+$. Therefore, the theorem is true for all positive integers n.
  - From this we conclude that $C = \mathbb{Z}_+$. Hence, the theorem is true for all positive integers n.
  - From this we conclude that $C = \mathbb{Z}_+$. Thus, the theorem is true for all positive integers n.
  - (ORIGINAL) From this we conclude that $C = \mathbb{Z}_+$, so, the theorem is true for all positive integers n.

  For such an elementary conclusion, saying anything but the last three sounds either pretentious or overly former (though it would not be incorrect). Note that *so* is a bit of a special case and can be used as a conjunction i.e. you do not need to start a new sentence.

No one likes to read repetitive language—not even mathematicians! So depending on the situation in may be possible to substitute *prove* with other verbs, such as: *show, exhibit, illustrate, demonstrate, use, apply, emphasise, clarify*. Though, we caution against using exotic verbs, such as *exemplify, attest, corroborate* etc.

2.2 Guidelines

In this Section we list common phrase-patterns used in mathematical exposition according to the purpose that they serve, e.g. to introduce a new definition, to draw a conclusion, to connect two pieces of an argument etc. Some of the phrases may appear synonymous, and in a lot of situations they may be interchangeable. Therefore, when you come to write a whole paragraph, you may wonder whether you should keep repeating your favourite phrase or whether you should always chose a different phrase if available? This is a matter of deciding between *parallelism* (or *simplicity*) and *elegant variation*, and we will discuss this stylistic dilemma in a future lecture. For now, pay attention to picking the appropriate phrase to use in a given situation.

The following examples may be trivial for native English speakers. However, the list itself should be of use to everyone. You should already be familiar with a vast majority of these patterns, as they are the building blocks of verbal and written mathematics; you will have heard them in lectures and seen them written down in textbooks and papers.

Within each heading the listed phrases are given according to approximate usefulness and frequency, though this is a personal preference. In general, you would do well to understand all of the phrases and be able to apply most of them accordingly.

Before proceeding with the list of phrase-patterns, we remark on three general principles that apply throughout.

2.2.1 Phrases that draw attention to people: do you need the pronouns?

Any phrase that contains pronouns referring to people—*I, you, one, we,* or *they*—draws attention to those people. For the purposes of this lecture, bear in mind that there is a slight difference between *We define* and *Define*, *They proved* and *It has been proved*, *One can show* and *It can be shown*, etc. In such cases, which phrase is preferred will depend on the context.

In a future lecture we shall discuss what authors should call themselves when writing and how to balance the two opposing tendencies: to draw attention to yourself and to stay neutral (either by using the imperative mood or the passive voice). Both extremes make for unpleasant reading.

2.2.2 The syntactic pleonasm: do you need *that*?

Which of the following statements is grammatically correct?

- Suppose X is an open set.
- Suppose that X is an open set.

Both are correct. The conjunction *that* is pleonastic, meaning it is redundant, though it can be included to improve the flow of the sentence.

However, the word *that* is not always a conjunction, so do not add or remove it arbitrarily unless you know what you are doing.

A. GOOD: We prove the theorem by induction.

B. BAD: We prove that the theorem by induction.

C. GOOD: We prove that the theorem holds when X is nonempty.

D. GOOD: We prove the theorem holds when X is nonempty.

Loosely speaking: *that* is a conjunction if it separates two (or more) verbs. In options C, it separates the verbs *prove* and *holds*. In option A, there is only one verb. Conversely, if there are two verbs at play, like in option D, you may be able to insert *that* to improve readability.

2.2.3 The dummy pronoun: what does *it* refer to?

The word *it* is a pronoun often used in mathematics to refer to an object under discussion.

The function is smooth. Therefore <u>*it*</u> is continuous.

Here *it* refers to *function*. But what about in these examples?

- From the preceding theorem, <u>*it*</u> follows that...
- However, <u>*it*</u> does not follow that we can show...

Is *it* referring to the *preceding theorem*? Or to something else in the second example?

The answer is that here *it* is a *dummy* pronoun, a pronoun which refers to nothing, but is part of the fixed phrase. Here are some other phrases that have a dummy pronoun:

- It is raining cats and dogs. (No math paper should have this phrase.)
- It is important to note...
- It clear...
- It was shown...

The last example is a construction in the passive voice and should be contrasted to the active voice *We showed...*, as discussed in Section 2.2.2.

What matters is that you do not want to be caught mixing up your pronouns. (This is similar to not using *one* in a phrase and as a number; see Section 1.2.8 in Lecture 1.) If in doubt, repeat the name of the object for which your pronoun stands.

- BAD: The set X satisfies the hypothesis. It follows that it is nonempty.
- GOOD: The set X satisfies the hypothesis. It follows that X is nonempty.

With those three general comments out of the way, we proceed to discuss some common phrase-patterns.

2.2.4 Defining or assuming

- *Let... be...* is used to introduce a definition simply and cleanly. The word *Let* is immediately followed by the symbol that is being defined.
  - Let f be a smooth function.
- *(We) define... to be...* is used to emphasise that a new object or concept is being defined.
  - We define a real-valued function to be *smooth*, if...
  - Define f to be a real-valued function with the following properties.
- *We say that...* is used to introduce a short definition.
  - We say that a set X is *nonempty*, if...
- *Suppose (that)...* is used to set up one branch of an argument that will later be contrasted to another branch or to set up an assumption at the beginning of a proof.

- Suppose X contains the element a. (Elsewhere: Suppose X does not contain the element a.)
- Suppose f is differentiable on $(0, 1)$. Then...
- Suppose that our argument holds in the special case...

- *Assume...* is used the same way as *suppose*.
- *Consider...* is used to highlight an aspect of an already defined object or to divert the argument towards a new idea.
  - Now consider the set X as a subset of this new set Y. (Elsewhere, previously: We define X as a subset of Z, etc.)
  - Let us consider all function satisfying property P.
- *Without loss of generality...* is used to indicate that there may be a choice involved, but that this choice is irrelevant for the argument at hand. For example, suppose that we are proving a topological result about a circle embedded in $\mathbb{R}^2$. Then:
  - Without loss of generality let C be the unit circle given by $x^2 + y^2 = 1$.

  Note that the phrase *without loss of generality* is actually saying the following: it can be shown that the choice we are making does not affect our argument. Hence, this phrase is used only in standard situations where the reader is already convinced that *it can be shown* is a valid statement, either because the statement is an elementary observation at their level of mathematics or because the statement has been shown elsewhere in the same piece of work.

2.2.5 Setting up and concluding

- *If..., then ...* is a direct translation of the the implies logical operator $\Rightarrow$. Indeed, the following two pieces of maths writing are equivalent:
  - If P, then Q.
  - $P \Rightarrow Q$.

  Note that in spoken English, the sentence:

  > If you pay for the sandwich, (then) you may take it out of the store.

  also carries the understood meaning:

  > If you do **not** pay for the sandwich, (then) you may **not** take it out of the store.

  Put differently, what the salesperson is telling the customer is:

  > You may take the sandwich out of the store *if and only if* you pay for it.

However, in mathematical English the *If. . . , then . . .* phrase is very clearly an implication in one direction!

NOTE: For the purposes of this course, and as a strong personal preference, this phrase will **always** take a comma before the word *then*.

- *if and only if. . .* is a direct translation of the equivalence logical operator $\iff$. Sometimes also found in the form of *a necessary and sufficient condition.*
- *From. . . we have/we conclude that. . .* is used to draw a conclusion from a previously proved statement. For example:
  - From Lemma 3.4 we have that S is the power set of T.
  - From the preceding argument, we conclude that. . .

2.2.6 Declaring intent

- *We prove/show. . .* is a fundamental way to state your intentions. In most proofs you will need a variation of these words, whether to indicate the type of proof or to indicate a step along the way.
  - Please prove the theorem using induction.
  - We prove that X is a nonempty set.
  - Now we can prove that the statement holds in the following cases. . .
  - Let us develop the necessary methods.

  The words *show* and *prove* can often be used as synonyms. However, direct word substitution does not always work.
  - GOOD: We prove the theorem.
  - BAD: We show the theorem.
  - GOOD: We show the theorem holds when f is the sine function.

2.2.7 Concluding

- *We conclude. . .* is used to signal a conclusion drawn from a chain of reasoning or an argument, like in the example. It is pretentious to conclude a statement holds if the consequence is well-known in the context or nothing has been shown.
  - GOOD: After applying the same argument we used in the proof of the Lemma, we conclude that X is a metric space with the desired properties.
  - BAD: If f is a smooth function, we conclude that f is continuous.
  - BAD: Let f denote a smooth function. We conclude that f is continuous.

  In the last two examples, *then* would have been more appropriate.

- GOOD: If f is a smooth function, then f is continuous.
- GOOD: Let f be a smooth function. Then f is continuous.

There exists an informal gradation between seemingly synonymous phrases that indicates the number and importance of arguments leading to the conclusion. For example, between *we conclude*, which is indicates more important arguments, and *then*, which indicates an immediate conclusion, there are middle steps: *implies* and *it follows*.

- *Implies...* is used to draw a direct conclusion from a preceding statement, more formally than *then*. Like the others, but perhaps most obviously so, the phrase is taken to be the word-equivalent of the logical operator $\Rightarrow$. Therefore, as with the logical operator, you must include both operands (e.g. you would never write $\Rightarrow Q$, you would write $P \Rightarrow Q$).
  - GOOD: The preceding theorem implies that G is a finite group.
  - BAD: Implies that G is a finite group.
  - GOOD: The fact that G is a finite group implies G cannot contain an element g of infinite order.
- *It follows that...* has the same meaning as *implies* and is used similarly.
  - GOOD: From the preceding theorem, it follows that G is a finite group.
  - BAD: We have shown that G is a finite group. From this it follows that it cannot contain an element g of infinite order.

  The second example contains at least two issues: it is wordy (*From this it follows that*) and it contains two instances of *it*, of which the second is ambiguous. See Section 2.2.3 for a discussion of *it* as the dummy pronoun.
- *Because, as, since, for...* are (subordinating) conjunctions that introduce the reason that leads to a certain conclusion.
  - *Because* emphasises the reason. This is the iron-clad conjunction that links the cause with the consequence.
  - *As/since* assumes the reason is well-understood or has been explained thoroughly and therefore does not need to be emphasised.
  - *For* introduces the reason as an aside. It is more "poetic" and can be confused with the preposition in phrases such as: For X a metric space,...

  Unless you are feeling particularly poetic, archaic, or confident, we would recommend *not* using *for*.
- *Accordingly, consequently, therefore, hence, thus, so...* are adverbs that introduce the conclusion (rather than the reason for the conclusion). The longer the adverb the more attention it will draw to itself, and the more weight it will give to the conclusion you are drawing.

Note that *so* can be used as a conjunction i.e. you do not need to start a new sentence.

2.2.8 Giving examples

- *For example...* is the most common way to introduce an example.
  - The theorem applies to topological spaces with fundamental group $\mathbb{Z}^2$ (for example, the torus).
  - For example, we could use Theorem 5 to study the torus.
- *One example of...* requires you to first recall the statement that you wish to discuss, then introduce an example of that statement.
  - One example of a manifold with fundamental group $\mathbb{Z}^2$ is the torus.

  That may sound a bit clunky, because you have to wait for the verb. So a neater statement would be:
  - The torus is one example of a manifold with fundamental group $\mathbb{Z}^2$.
- *e.g.* is an abbreviation for the Latin expression *exempli gratia* meaning *for example*. Some sources would recommend using the English expressions over the Latin ones, though it is a matter of preference especially in less formal texts such as these lecture notes. In formal papers, it should be avoided or used sparingly. The phrase is written at the beginning of a parenthesis or it is set off by commas.
  - The theorem applies to topological spaces with fundamental group $\mathbb{Z}^2$ (e.g., the torus).
  - The theorem applies to topological spaces with fundamental group $\mathbb{Z}^2$, e.g., the torus.

  You may have noticed that these notes drop the comma after the phrase—that is technically not correct but writing a comma after the period just looks abysmal (so we've taken this small liberty of omitting it here).
- *In particular/Particularly, Specifically...* are all used to zoom in on an aspect of whatever is being discussed, including examples.
  - The theorem applies to topological spaces with fundamental group $\mathbb{Z}^2$. In particular, a special case is the torus.
  - The theorem applies to topological spaces with fundamental group $\mathbb{Z}^2$. Specifically, it applies to the family of manifolds obtained by...

2.2.9 Structuring a longer argument

Arguments longer than a few lines should be structured using keywords that help orientate the reader with respect to the "bigger picture". What are you proving first? What comes next? Here are the common phrases used at the start of a new idea.

- *First/Firstly, secondly, thirdly, finally...*
- *First, then/next, finally...*
- *To begin with... then...*
- *Subsequently, now, after, later...*

2.2.10 Calling upon arguments

You will often need to invoke arguments that are not immediately present, either your own (from somewhere within that same piece of writing) or someone else's (from a book or paper). Here are some common phrases that will serve your purpose.

- *In Section/Theorem/Lemma... we prove/it was proven...*
- *Exercise/Example/Proposition... proves that ...*
- *Recall (that)...*
- *In the discussion above, In the discussion below, In the following ...*

2.2.11 Emphasising and repeating oneself

Theorems, propositions, lemmas and the like are emphasised through their special, labelled status. However, in ordinary text, such as proofs and exposition, you may wish to emphasise certain statements on a local level to make sure the reader does not miss a particularly salient point. You can do this in two ways: by using certain keywords (*actually, obviously, etc.*) or by repeating yourself (*indeed, in other words, etc.*).

- First, let us discuss the keywords *actually* and *really*. They are never strictly necessary, and should not be inserted arbitrarily, but only in places where the reader is likely to be surprised by the result or the result must be emphasised. For example:
  - Despite the function's abstract definition, we have actually shown it to be homotopic to the identity.
  - Actually, the simplest way to disprove the result was to find a counterexample.
- Next, let us discuss the following set of phrases:
  - *Certainly, Surely...*

- *Clearly/It is clear that...*
- *Obviously/It is obvious that...*
- *Trivially/It is trivial...*

These phrases should be used with caution. Depending on who is reading your work, you may come across either as condescending or as displaying a lack of willingness (or ability!) to work out the details. Consider these two examples.

A. Define the set X to have one element. It is then obvious that the set is nonempty.

B. Clearly, a manifold whose fundamental group is perfect has a trivial first homology group.

If you are speaking to a seven-year-old, Statement 1 may be of the appropriate level and is a repetition that highlights a different way of saying nonempty. A twelve-year-old might already find it condescending. And no such statement should ever make it into a university-level maths paper.

Statement 2 comes across as condescending to anyone who is not familiar with the specialised terms. In contrast, those who are familiar with the terms would consider this a fact that does not need stating but can be directly applied. When the group of mathematicians stuck in the middle—students learning the terms—write a statement like this, they just come across as regurgitating a fact they do not quite understand.

Before writing any of these phrases you should ask yourself the following questions:

> If the statement is obvious, do I need to make the statement at all?
> If I *do* need to make the statement, then is it actually obvious?
> Should I perhaps be giving a proof after all?

- Lastly, let us consider how you can repeat yourself to good effect.
  - *Indeed,...* is used to rephrase what you have just said in a way which will make it easier to draw a specific conclusion or move on with the chain of the argument.

    A. GOOD: Certain phrases should be used with caution as they may be deemed condescending. Indeed, you should first ask yourself whether they are necessary.

    B. BAD: Certain phrases should be used with caution as they may be deemed condescending. Indeed, they may be deemed condescending.

    C. BAD: Certain phrases should be used with caution as they may be deemed condescending. Indeed, other phrases may not be deemed condescending.

Statement 1 balances repetition with a new aspect of the same statement. (Is it necessary to risk sounding condescending?)

Statement 2 repeats what was already said.

Statement 3 repeats the obvious negative of what was already said.

– *In fact,. . .* is used the same way as *indeed*.

– *In other words, . . .* likewise.

– *That is,. . .* likewise. In spoken English you may hear this phrase used in its expanded version *that is to say*.

> The intersection of these two open sets U and V in $\mathbb{R}^2$ is nonempty, that is, we can find a point $x \in U \cap V$ and an open set Z such that $x \in Z$ and $Z \subset U \cap V$.

– *i.e.,. . .* is an abbreviation for the Latin expression *id est* meaning *that is*. Even more so than with *e.g.*, in formal writing it is hard to make a case for using the Latin equivalent of the handy English equivalent *that is*. The phrase should either be in parenthesis or set off by commas.

2.2.12 Universal quantifier: all, any, every, each

The symbol for the universal quantifier $\forall$ is usually translated into words as one of the following:

> *all, any, every, each, for all, for any, for every, for each, for any choice of, given any*

Which phrase you use will depend on the sentence structure, but here are a few pointers.

- Suppose $\mathfrak{F}$ is a subset of continuously differentiable functions that has some property P (for our purposes it does not matter which set this is and what the property is). Suppose that you wish to say that you can prove all functions in $\mathfrak{F}$ also have property Q. You could do it in numerous ways:

  – For each function $f \in \mathfrak{F}$, we can show that f also has property Q.

  – We can show that every function in $\mathfrak{F}$ also has property Q.

  – For all functions in $\mathfrak{F}$, we can show that f also has property Q.

  – Given any function $f \in \mathfrak{F}$, we can show that f also has property Q.

- Consider that last example. There, the word *any* can be confusing. In non-mathematical English, you may hear people saying:

  > Do you have any money? Yes, I have some.
  > Pass me any of those pencils. (I don't care which one.)

  In the first case, *any* refers to a nonzero quantity; in the second case, *any* refers to a choice, where the choice is unimportant and this can cause confusion in mathematics. Take the following example.

  - Show that any differentiable real-valued function attains a maximum on any closed interval.

  Here it would be incorrect to pick out one function, say $f(x) := x$, and one interval, say $[0, 1]$, and show that the result holds. When using *any* as shown, a mathematician means the result has to be shown *for all* such function *on all* such intervals. This is a common abuse of the word *any* and it is part of standard maths writing.

- Consider next the words *each* and *every*. They are mostly interchangeable, though they may emphasise different parts of a statement. *Each* emphasises the individual object and sets up the expectation that something will be done with that object. *Every* emphasises a group property that this object has.

  When the group property is emphasised, using *every* or *all* is better.

  - GOOD: Every smooth function is differentiable.
  - GOOD: All smooth functions are differentiable.
  - BAD: Each smooth function is differentiable.

  When the individual object is emphasised *every* or *each* can be used:

  - GOOD: To each $f \in \mathfrak{F}$ we can assign a natural number.
  - GOOD: To every $f \in \mathfrak{F}$ we can assign a natural number.

  However, note that if there are only **two** objects, for example when you are colouring vertices on a graph either black or white, you must use *each*:

  - GOOD: To each colour we assign a subset of vertices.
  - BAD: To every colour assign a subset of vertices.

Problem Sheet 2

Suppose the statement of each problem is part of your written work. You now have to write down the next statement.

- Choose from amongst the options the single best followup line. There is always only one correct answer.
- Why are the other options incorrect?

Make sure to consider material from both Lectures 1 and 2.

PROBLEM 1. Let us show that f is smooth.

A. Show f is continuous.

B. From f is continuous we will conclude the result after some more work.

C. We first prove that f continuous.

D. We begin by proving that f is continuous.

E. You need that f continuous.

PROBLEM 2. *Proof.*

A. Let a smooth function be f.

B. Let f be a smooth function.

C. Define a smooth function to be f.

D. Define f.

PROBLEM 3. Define the sets $X := \{0, 1, a\}$ and $Y := \{0, 1, 2, 3\}$, where a is some integer.

A. If $a = 2$, then $X \subset Y$.

B. Only if $a = 2$, then $X \subset Y$.

C. If $X \subset Y$, then $a = 2$.

D. The following is true: $X \subset Y$ if and only if $a = 2$.

E. Let us conclude that if $a = 2$, then $X \subset Y$.

F. If $a = 2$ then $X \subset Y$.

PROBLEM 4. We have shown that G is a finite cyclic group.

A. It follows that it is not isomorphic to $\mathbb{Z}$.

B. It follows that G is not isomorphic to $\mathbb{Z}$.

C. It can be shown that it is not isomorphic to $\mathbb{Z}$.

D. G is not isomorphic to $\mathbb{Z}$.

E. We conclude that G is not isomorphic to $\mathbb{Z}$.

F. From this we conclude that G is not isomorphic to $\mathbb{Z}$.

G. Implies G is not isomorphic to $\mathbb{Z}$.

PROBLEM 5. We have shown that G is a finite cyclic group.

A. Clearly, there exists a number $m \in \mathbb{N}$ such that $|G| < m$.

B. It is obvious that the order of G is bounded by some number m.

C. Indeed, G is finite.

D. Let $m \in \mathbb{N}$ be a number such that $|G| < m$.

E. Ie the order of G is bounded by some number m.

LECTURE 3

Troublesome Words

English words have been successfully misunderstood for hundreds of years. Hamlet, Shakespeare's great tragic hero, is notorious for keeping his interlocutors guessing.

HAMLET:	Words, words, words.
POLONIUS:	What is the matter, my lord?
HAMLET:	Between who?
POLONIUS:	I mean the matter you read, my lord.

(*Hamlet*, 2.2 192–195)

Here Polonius asks *What is the matter?* and intends it as a question about reading material. Hamlet (deliberately) misunderstands the question as the fixed expression that is used to enquire when something is amiss. Such wordplay is rare in mathematics, but if you are not careful which words you use, the misunderstandings can be equally mind-boggling.

In this lecture we cover certain words that can lead to confusion if misused. In Section 3.1, we discuss *determiners*—words that precede nouns and give them context. In Section 3.2, we discuss potentially ambiguous pronouns (*it, they*) and words that require thorough justification (*problem, interesting*).

3.1 Determiners

In language theory, a *determiner* comes before a noun and provides the noun with context. The best-known examples are the definite (*the*) and indefinite (*a*, *an*) articles, which tell the reader or listener whether a particular noun is familiar or not. Other determiners also commonly appear in mathematical English and getting them wrong can lead to confusion or parsing difficulties.

- The zero article: when an article is not needed.

  ~~The~~ Precision is important.

- The negative article: *No.*

  <u>*No*</u> topological space satisfies these conditions.

- Possessives: *its* and *their*

  For each function in the sequence (f_i), compute <u>*its*</u> derivative.
  Next, consider the groups G and H and <u>*their*</u> associated sets X_G and X_H.

- Quantifiers: *all, any, each, every, both, several, some, many.* In Section 2.2.12 we discussed the the first four quantifiers. The others should be familiar and unambiguous: *both* indicates two object; *several* indicates more than two objects but not many; *some* indicates an unspecified number; *many* a large number.

3.1.1 The indefinite articles: *A* or *An*

The rules governing articles are the same as in standard English:

- *A* precedes a singular noun that begins with a consonant sound, even if it is written with a vowel (so: <u>*a*</u> *function*, but also: <u>*a*</u> *unit ring*).

- *An* precedes a singular noun that begins with a vowel sound, even if it is written with a consonant (so: <u>*an*</u> *eigenvalue*, but also: <u>*an*</u> *NP-complete problem*).

3.1.2 Definite vs indefinite article

Use *the* if the reader knows which specific object you are referring to, and *a/an* if the reader does not.

Here are a few cases where the definite article is necessary.

- If the object was just defined:

  > Let $F(x) := x^3 - 2$. <u>*The*</u> function is negative on...

  Or if the object was defined earlier:

  > Recall that <u>*the*</u> function F is negative on...

- If the object in question is known and understood to be unique in the context:

  > We prove that g_0 is actually <u>*the*</u> identity element of the group and that g_1 is <u>*the*</u> inverse of g_2.

  Note that the use of *the* in front of *group* implies the group was defined earlier.

- If the object is defined after the article has to be chosen, then using *the* is justified if the object will be uniquely determined by the definition. For example:

  > Let X be <u>*the*</u> subset of $\mathbb{R}^2$ obtained by removing all points with at least one rational coordinate.

  However, in the following we cannot use *the*, as the object we are defining is not unique. Instead, we need *a*, or *any*, or some other existential determiner.

  > Let Y be <u>*a*</u> subset of $\mathbb{R}^2$ containing the point $(0, 0)$.

Here are a couple of examples when you should use the indefinite article:

- If you wish to discuss the properties of a general object, that is, of an object indistinguishable from other objects of its type:

Let n be <u>*a*</u> natural number and S <u>*a*</u> set of natural numbers.

You could also write *some* or *any* instead of *a*, if you wanted to emphasise the choice was arbitrary. The meaning is identical; the indefinite article just draws the least attention to itself.

- If you have shown that something is an object of a certain type:

  We have shown that M is <u>*a*</u> manifold.

  You could not use any other determiner here.

Most importantly, singular countable nouns cannot be used alone in a sentence; they always require a determiner (*the* function, *a* function, *any* function, *every* function, etc.).

3.1.3 Plural nouns and the zero article

The zero article refers to the situations when nouns take no article. Two cases stand out:

- uncountable nouns, such as *air*, *patience*, *work*, when talked about in general:

  Even mathematicians need ~~the~~ air to breathe.
  ~~The~~ Patience is key.

- plural countable nouns, such as functions, sets, mathematicians, when talked about in general (even ~~the~~ mathematicians need air to breathe):

  Which ~~the~~ functions does the paper study?
  I hate ~~the~~ categories. I love working with ~~the~~ groups.

Note that plural countable nouns do take the definite article if they have been introduced or specified before:

<u>*The*</u> functions we discussed yesterday are pretty cool.
Let us now return to <u>*the*</u> two groups, G_1 and G_2, defined above.

3.1.4 The negative article: *no*

There are different ways to negate a statement. Consider:

There exists a function satisfying this condition.

This statement can be correctly negated in any of the following ways:

A. There does not exist a function satisfying this condition.

B. There exists <u>*no*</u> function satisfying this condition.

C. <u>*No*</u> function satisfies this condition.

Option A negates the verb, whereas options B and C employ the emphatic negative article *no* to achieve the same result. Option C also removes the verb *to exist*, making for a cleaner, crisper statement, though one that de-emphasises existence.

Remark 3.1. Negative statements should contain a single negation like in our examples A, B, and C above. Positive statements should contain no negation.[1] Put differently, many statements can be logically negated and correctly translated into an English sentence with two negative words (e.g. *I understand* can become *I do not misunderstand*). Specifically, a typical maths example would be:

> All functions in $\mathcal{F}$ satisfy this condition.
> There exists no function in $\mathcal{F}$ that does not satisfy this condition.

Though the two statements technically have the same meaning, the first is preferred over the second (unless the function not satisfying the condition needs to be emphasised for some reason).

3.2 Ambiguous words

3.2.1 Ambiguous pronoun: *It*

It is the third person singular pronoun that carries no gender (unlike *he* and *she*). Therefore, instead of having to repeat the name of a mathematical object that was just introduced, we often use *it* because it's shorter and more elegant.

> If the group is finite, then *it* cannot have an element of infinite order.

The *antecedent* of a pronoun is the noun which the pronoun replaces. So in our example, the antecedent of *it* is *group*. However, sometimes the antecedent is not clear and this gives the reader unnecessary pause.

- We are in the middle of the proof and we come to this sentence:

  > If X is finite, Y is infinite, and $X \cap Y = \emptyset$, then *it* is empty.

  Without context we are lost. However, even with context what the *it* refers to can be unclear. Is it referring to one of the two sets or to something else previously named? In the former case, this is outright bad writing. In the latter case, it may be possible to understand what is going on, but it is not advisable to let your pronoun refer too far back, especially if you have other potential candidates for the antecedent much closer by (such as X and Y).

- We are in the middle of an exposition and we come across this sentence:

  > The space does not satisfy the main condition of the Lemma, therefore we must modify *it* before proceeding.

  The problem is essentially the same here, but a little harder to spot. What needs to be modified? The *space*, or the *main condition*, or the *Lemma*? The antecedent could be any of those nouns. It may be apparent to you, as the author of a sentence, which antecedent you mean, but your reader will not know.

[1] Does this statement follow its own advice? Do not think too hard about this question.

The situation is even more complicated that you think, because the word *it* also has a possessive form, *its*, and the word *it* often occurs in the phrase *it is*. The following statement is grammatically correct and conveys the correct meaning but is hard to parse:

> The possessive form of the word *it* is *its* and *it* can be confused with *it's*, the contraction of the words *it is*.

Let us unpack this statement. There are four things to worry about here:

- *It* is a pronoun, which can cause confusion as shown above. Can you guess what the second *it* in the statement refers to?[2]
- *Its* is the possessive of *it*. So instead of saying: *The function's derivative is positive,* it can sometimes be useful to say: *Its derivative is positive.*
- *It is*, contracted to *It's*, occurs frequently in written English, though it should not appear at all in formal mathematical writing.[3] However, *it's* looks like the correct possessive form of *it* because of other possessives that end in *'s*, such as *Riemann's*, and can therefore remain unnoticed (and uncorrected).
- *It* occurs as a dummy pronoun in passive constructions such as *it can be confused* and in phrases such as *it follows*; see Section 2.2.3 in Lecture 2, for more on the dummy pronoun.

After this, how can one ever write *it* and get it right? Here is an example of clear writing which does get it right. It comes from §11 on The Maximum Principle from *Topology* by James R. Munkres. Context helps, but note how, in addition to using *it* prodigiously, Munkres also keeps repeating his nouns (*element, box*) so the reader is at no point confused what he is referring to.

> One can give an intuitive "proof" of the maximum principle that is rather appealing. It involves a step-by-step procedure, which one can describe in physical terms as follows. Suppose we take a box, and put into it some of the elements of A according to the following plan: First we pick an arbitrary element of A and put it in the box. Then we pick another element of A. If it is comparable with the element in the box, we put it in the box too; otherwise, we throw it away. At the general step, we will have a collection of elements in the box and a collection of elements that have been tossed away. Take one of the remaining elements of A. If it is comparable with everything in the box, toss it in the box, too; otherwise, throw it away. Similarly continue.

Do you understand what each of the highlighted pronouns stands for? Now, consider what would have happened had Munkres not bothered with the pronouns and instead repeated the nouns. Would the explanation not have been even clearer? Judge for yourself.

[2] The second *it* refers to the *its*. Yes, it is discombobulating.
[3] *It's* did creep into these Lecture Notes once above. Kudos to everyone who spotted it!

> One can give an intuitive "proof" of the maximum principle that is rather appealing. The proof involves a step-by-step procedure, which one can describe in physical terms as follows. Suppose we take a box, and put into the box some of the elements of A according to the following plan: First we pick an arbitrary element of A and put the element in the box. Then we pick another element of A. If the element is comparable with the element in the box, we put the element in the box too; otherwise, we throw the element away. At the general step, we will have a collection of elements in the box and a collection elements that have been tossed away. Take one of the remaining elements of A. If the element is comparable with everything in the box, toss the element in the box, too; otherwise, throw the element away. Similarly continue.

Much worse, is it not? Do you understand why? Compare side by side a sentence from the original (A), the same sentence from the second version without pronouns (B), and a third version which labels the elements symbolically (C).

A. If it is comparable with the element in the box, we put it in the box too; otherwise, we throw it away.

B. If the element is comparable with the element in the box, we put the element in the box too; otherwise, we throw the element away.

C. If X is comparable with Y, we put X in the box too; otherwise, we throw X away.

Such a comparison makes it clear that the pronoun is being used a temporary label to distinguish the element spoken about (the one we "hold in our hand", if you will) from the element that has already been put in the box; the first we call *it*, while the second retains the full noun-label *the element*. Munkres is able to do be clear because he carefully maintains the use of *it* throughout that sentence.

Finally, observe that whilst *it* changes antecedents often, it always refers to the same noun within each sentence. This is not a rule, and it does not always have to be true, but it may be a useful guideline.

3.2.2 Ambiguous pronoun: *They*

They is the third person plural pronoun. Remarks similar to those given in Section 3.2.1 apply here as well. There must never be any doubt about the antecedent of the pronoun. In the following example, it is not clear what *they* is.

> Let us now consider the functions f_1 and f_2 and their first derivatives. We wish to prove that <u>*they*</u> are continuous.

If *they* is standing in for *their first derivatives*, then this fact should be made explicit.

> Let us now consider the functions f_1 and f_2 and their first derivatives. We wish to prove that <u>*their first derivatives*</u> are continuous.

But this version is repetitve. Note that once the objects (*their first derivatives*) have been introduced, even if you have to repeat the name of the objects, you can do so with precision by using the definite article (*the*) and the noun (*derivatives*), and you can avoid any other delimiters (*their*, *first*, and five others, were there others).

> Let us now consider the functions f_1 and f_2 and their first derivatives. We wish to prove that *the derivatives* are continuous.

Regardless of this simplification, the sentence still appears clumsy and repetitive, which indicates that it can probably be neatened further. So take a moment to neaten it.

> Let us now consider the functions f_1 and f_2. We wish to prove that *their first derivatives* are continuous.

3.2.3 Ambiguous demonstratives: *This, these, that, those*

There are four so-called *demonstrative* pronouns: *this* and its plural *these*, which denote nearer objects, and *that* and its plural *those*, which denote more distant objects. How near something is, physically or temporally, depends on the point of view and context. For example, suppose that you believe the problem you have just been describing to a colleague is more interesting than the one the professor described yesterday. You might say:

> *This* is interesting to think about; *that* is not.

What you would actually mean is:

> *This* (problem I described to you) is interesting to think about; *that* (problem the professor described yesterday) is not.

In speech, such omissions are acceptable and cause no misunderstandings, but they make for sloppy writing.

In theses, papers, and PhD applications you will often be tempted to write statements such as:

> *This* is an interesting problem to study.
> *That* is the reason I would like to study *this*.

Unless you have specified explicitly which problems and what reasons, such statements are ambiguous. (The second statement is also just bad writing: in few circumstances should you be juggling the pronouns *this* and *that* in the same sentence). Furthermore, the words *interesting, problem*, and *reason* are often ambiguous in their own right. We consider these words in the next section.

3.2.4 Words requiring justification: *problem, interesting, reason*

The inclusion of some words in writing calls for more careful justification than the inclusion of others. You personally may find a topic *interesting* or *exciting*, but just stating your feelings will not get anyone else interested or excited. The following statement is no good:

> This is an *interesting* problem to study.

Indeed, such a statement ought to be followed by a thorough explanation:

> This is an *interesting* problem to study because... [insert a specific historical, contextual, mathematically intriguing reason].

Another word to watch out for is *problem*. Courses have *Problem Sheets*, meaning lists of mathematical exercises. However, the word is used in other contexts, such as: *theoretical problems* and *applied problems*, and *the problems we encountered while applying a method*. With this in mind, let us once more return to the example above:

> This is an interesting *problem* to study.

Part of making this sentence work is giving meaning to the word *interesting*, but the other part is making *this* and *problem* work together and "point" towards something which can genuinely be called a specific problem. Take the following example:

> The critical set of a Morse-Bott function $f\colon M \to \mathbb{R}$ is a submanifold of M. This is an interesting problem to study.

The two sentences are not connected because no problem has been stated in the first sentence. In fact, most problems require a lot of additional information to be justified as relevant. The following example has bogus mathematical content, but vaguely approximates the form of explanation you are expected to offer.

> The critical set of a Morse-Bott function $f\colon M \to \mathbb{R}$ is a submanifold of M, called a critical submanifold. Restricting to functions $f \in \mathfrak{F}$ on manifolds of dimension $k \geq 5$, one may ask whether it is possible to determine the weak homotopy type of any such critical submanifold using the method of Shishkin [Sh07]. This is an interesting problem because Repin and Levitan [RL09] showed that for functions f belonging to the smaller class $\mathfrak{F}_0$, any such critical submanifold is contractible. Moreover the example $f(x) = \log \sin^2(x)$ from above shows that their result cannot be expected to hold verbatim in our setting...

Similar comments apply to other words praising a piece of mathematics, such as *nice, beautiful, significant, fun.* Most other adjectives that convey your opinion should also be rigorously questioned before written down:

- This is a *quick* method.

  Compared to what method and how do you measure speed?

- This is a *complicated* argument.

  Compared to which other argument and how do you measure complexity?

- This is *painful* computation.

  Compared to whose level of pain while doing what other computation?

These and similar phrases are used amongst mathematicians on a daily basis as an abbreviated way of conveying opinion within context (they are part of the "math lingo"). However, on the page that context needs to be made abundantly clear because the reader is removed from the writer's presence.

Lastly, the word *reason* also requires a clear antecedent. Recall our example from Section 3.2.3:

> <u>*That is the reason*</u> I would like to study...

The danger is that one ambiguous statement leads to another. So when asked why you wish to work in a certain area of maths or work on a certain problem, the answer should not be *because it is <u>interesting</u>.* And why is interesting? *Because it is <u>important</u>.* And why is it important? *Because it is the <u>reason</u> I got into mathematics!* We will return to this in a future lecture on writing applications, but the moral is applicable in any math-related context where you wish to use a slippery label that expresses an opinion. The justification for using the label should be grounded in fact: mathematical, historical, or personal (specifically, your intellectual qualifications for the project).

Problem Sheet 3

In Problems 1–5 choose the statement that has the appropriate article (or lack thereof) before every noun. No maths is involved. There is only only one correct answer in each problem.

Problem 1.

A. Let us recall the definition of a general linear group $GL(n, \mathbb{C})$.

B. Let us recall the definition of the general linear group $GL(n, \mathbb{C})$.

C. Let us recall definition of the general linear group $GL(n, \mathbb{C})$.

D. Let us recall the definition of general linear group $GL(n, \mathbb{C})$.

Problem 2.

A. Let U_1 and U_2 be an unitary matrix.

B. Let U_1 and U_2 be the unitary matrix.

C. Define U_1 and U_2 to be the unitary matrices.

D. Let U_1 and U_2 be an unitary matrices.

E. Let U_1 and U_2 be unitary matrices.

Problem 3.

A. In our paper, we study the entropy of a thermodynamic system.

B. In our paper, we study entropy of a thermodynamic system.

C. In our paper, we study an entropy of a thermodynamic system.

D. In our paper, we study a entropy of thermodynamic systems.

Problem 4.

A. Suppose M is m-dimensional manifold.

B. Suppose M is a m-dimensional manifold.

C. Suppose M is an m-dimensional manifold.

D. Suppose a manifold M is m-dimensional.

Problem 5.

A. I understood theorems we covered in the Analysis lecture yesterday.

B. I understood the theorems we covered in Analysis lecture yesterday.

C. I understood the theorems we covered in the Analysis lecture yesterday.

D. I understood the theorems we covered in an Analysis lecture yesterday.

PROBLEM 6. Consider the statement:

There exists an open set S such that $s_0 \in S$.

Which of the following is a correct negation of the statement using the negative article? There is one correct answer. No maths is involved.

A. There no does exists an open set S such that $s_0 \in S$.

B. There does no exist an open set S such that $s_0 \in S$.

C. There no exists an open set S such that $s_0 \in S$.

D. There exists no open set S such that $s_0 \in S$.

E. There exists an open set S no such that $s_0 \in S$.

F. There exists an open set S such that no $s_0 \in S$.

In the following excerpt from Munkres's *Topology* pay attention to the pronouns. What are their antecedents? We have introduced the numbering in square brackets for ease of reference.

Theorem (Well-ordering theorem). *If A is a set, there exists an order relation on A that is a well-ordering.*

This theorem was proved by Zermelo in 1904, and it [1] startled the mathematical world. There was considerable debate as to the correctness of the proof; the lack of any constructive procedure for well-ordering an arbitrary uncountable set led many to be skeptical. When the proof was analyzed closely, the only point at which it [2] was found that there might be some question was a construction involving an infinite number of arbitrary choices, that [3] is, a construction involving—the choice axiom.

Some mathematicians rejected the choice axiom as a result, and for many years a legitimate question about a new theorem was: Does its [4] proof involve the choice axiom or not? A theorem was considered to be on somewhat shaky ground if one had to use the choice axiom in its [5] proof. Present-day mathematicians, by and large, do not have such qualms. They [6] accept the axiom of choice as a reasonable assumption about set theory, and they [7] accept the well-ordering theorem along with it [8].

In each of the following eight problems, corresponding to the eight highlighted words, choose the one correct description of the word's role.

Problem 7. *This theorem was proved by Zermelo in 1904, and it [1] startled the mathematical world.*

A. *it* is part of a passive voice construction.

B. *it* is a pronoun referring to the year 1904.

C. *it* is a pronoun referring to the noun *theorem*.

Problem 8. *When the proof was analyzed closely, the only point at which it [2] was found that there might be some question...*

A. *it* is part of a passive voice construction.

B. *it* is a pronoun referring to the noun *point*.

C. *it* is a pronoun referring to the noun *proof*.

Problem 9. *...number of arbitrary choices, that [3] is, a construction...*

A. *that* is part of a passive voice construction.

B. *that* is a pronoun referring to the noun *number*.

C. *that* is part of a phrase pattern used to rephrase something that has been said.

Problem 10. *Does its [4] proof involve the choice axiom or not?*

A. *its* is a pronoun referring to the noun *theorem*.

B. *its* is a possessive determiner associated to the noun *theorem*.

C. *its* is a possessive determiner associated to the noun *question*.

Problem 11. *A theorem was considered to be on somewhat shaky ground if one had to use the choice axiom in its [5] proof.*

A. *its* is a possessive determiner associated to the noun *axiom*.

B. *its* is a possessive determiner associated to the noun *ground*.

C. *its* is a possessive determiner associated to the noun *theorem*.

Problem 12. *They [6] accept the axiom of choice as a reasonable assumption about set theory,...*

A. *They* is part of a passive voice construction.

B. *They* is a pronoun referring to the noun *qualms*.

C. *They* is a pronoun referring to the noun *mathematicians*.

Problem 13. *…and they [7] accept the well-ordering theorem*

A. *they* is a pronoun referring to the other pronoun *they*.

B. *they* is a pronoun referring to the noun *choice*.

C. *they* is a pronoun referring to the noun *mathematicians*.

Problem 14. *…accept the well-ordering theorem along with it [8].*

A. *it* is a pronoun referring to the noun *theorem*.

B. *it* is a pronoun referring to the noun *theory*.

C. *it* is a pronoun referring to the noun *axiom*.

Problem 15. Which of the following fragments contain potentially ambiguous or contentious opinion words.

A. This is a difficult problem

B. Group G

C. An interesting approach

D. A brilliant choice

E. We proceed to write down the proof in detail

F. We spent much time trying to prove

G. This is a famous lemma

H. This is a solid construction, which will give us the result we need

I. A continuous function

J. Their writing style is flowery

K. I love mathematics

L. I hate mathematics

M. I am God's gift to mathematics

N. I have a degree in mathematics from ETH Zürich

O. I have a degree in mathematics from the best university in the whole universe

LECTURE 4

Unfortunate Word Choices

In the previous lecture, we dealt with articles and ambiguous words in general. In this lecture, we continue to discuss troublesome words: in Section 4.1, we look at words that can be deleted and phrases that can be shortened; in Section 4.2, we look at English words with multiple meanings (standard and specialised), relative clauses (is it *which* or *that*?), pairs of commonly confused words (*advice/advise*, *then/than*), and give a brief overview of British versus American spelling. Both sections include extensive tables with examples, most of which should be familiar, but all of which we recommend that you review at least once.

4.1 Unnecessary words

Some words from spoken English make their way into writing where they clutter up sentences (e.g. *actually, really*). Others words come as part of longer phrases, where shorter phrases would have been preferred (e.g. *due to the fact that* instead of *since/because*). Cutting out unnecessary words speeds up any piece of prose, but particularly in mathematics it helps the reader get to the idea faster.

4.1.1 Emphatic words: *actually, really, very*

Similarly to the words *actually, really, obviously* that were covered in Section 2.2.11, the words *very, most, least* might be tempting to use for emphasis:

> This is a <u>*very*</u> interesting problem, investigated in a <u>*most*</u> insightful manner, though explained in the <u>*least*</u> helpful language.

However, they can almost always be deleted. In the first two cases, remove the word and adjust the indefinite article (see Section 3.1.1); in the last case, *least helpful* needs to be changed to *unhelpful* because just deleting *least* leaves an ungrammatical construction.

> This is a**n** interesting problem, investigated in a**n** insightful manner, though explained in **un**helpful language.

In fact, the words we have removed usually indicate the adjectives that are themselves questionable. Some serious justification would be needed before the words *interesting, insightful* and *unhelpful* could be included (see Section 3.2.4).

REMARK 4.1. A crude, mechanical method of checking your emphatic words is using a search function in your writing environment to find words that end with *-ly*. This search will highlight most adverbs (as most adverbs end with *-ly*), and especially the exotic ones, though it will miss the handful of common ones, including *very, most, least, never, often*. It will also catch other words that are not adverbs, such as *imply, rely* and *reply*. However, it is worth trying it on a few pieces of your writing, just to get an idea of where you stand with respect to adverbs. It can be eye-opening.

4.1.2 Verbose phrases

Certain phrases can be shortened. Consider Table 4.1: on the left are phrases that can be replaced by *because* (or its synonyms, e.g. *since* and *as*; see Section 2.2.7 in Lecture 2). In general, though, there are two ways to directly simplify a verbose phrase. We can either remove all but a single necessary word (Table 4.2), or we can replace the phrase with another equivalent word (Table 4.3).

The third column in each of Tables 4.1, 4.2, and 4.3 is the approximate number of hits one gets when searching www.arxiv.org for the particular string (with quote marks) given in the first column. The tables are not exhaustive; they are meant to illustrate the kinds of phrases you should be on the look out for.[1] As you can see from the numbers, on the one hand, there are thousands of cases where these verbose phrases could have been improved. On the other hand, even silly phrases that you think no one would ever write—*account for by the fact that*, *eliminate altogether*, and *an example of this is the fact that*—do appear in papers, if rarely. No one is immune; no phrase is too convoluted to be used by accident.

You cannot always automatically substitute a verbose phrase for a neater one. Be aware of the grammar and the nuances in meaning.

Here are some examples:

- BAD:

  > Note that this construction only works as a consequence of us having already established that X is a vector space.

- GOOD:

  > Note that this construction only works because we have already established that X is a vector space.

- BAD:

  > In this course we will not study 3-dimensional manifolds separately, despite the fact that they give rise to many real-world applications.

- GOOD:

  > In this course we will not study 3-dimensional manifolds separately, although they give rise to many real-world applications.

- BAD:

  > In the event that a set contains an element, we say that the set is *nonempty*.

[1] Have you spotted any of the long-winded phrases in these notes? Let us know via the forum!

due to the fact that	because	2.2M
in view of the fact that	because	89K
owing to the fact that	because	35K
for the reason that/for this reason	because	17K
on account of	because	11K
on the grounds that	because	2K
the reason is because	because	260
accounted for by the fact that	because	169
based on the fact that	because	46

Table 4.1: Some phrases that can be replaced with *because*.

in order to	to	3.5M
by means of	by	212K
for the purpose of	for	145K
during the course of	during, while	32K
as to whether	whether	15K
connected together	connected	2K
the question as to whether	whether	2K
alternative choice	choice	1K
equal to one another	equal	502
fewer in number	fewer	328
assemble together	assemble	169
collaborate together	collaborate	63
eliminate altogether	eliminate	19

Table 4.2: Some phrases can be reduced to a single word.

a number of	some	351K
despite the fact that	although	87K
a small number of	a few	72K
in connection with	about, concerning	34K
in the event that	if	17K
take into consideration	consider	10K
in the vast majority of cases	usually	1K
make an assumption that	assume	1K
an example of this is the fact that	for example	3

Table 4.3: Some phrases can be reduced to a single word.

- GOOD:

  If a set contains an element, we say that the set is *nonempty*.

- BAD:

  The formalism is designed to give you the correct answer in the vast majority of cases.

- GOOD:

  The formalism is designed to usually give you the correct answer.

4.2 Unfortunate word choices

All writers make unfortunate word choices in their first drafts, not because they do not know their words, but because they are neither gods nor machines. And whilst gods may aid with inspiration and machines may aid with spellchecks, ultimately *you* will still need to proofread your own work (or get an equally knowledgeable friend to do it!).

Here are the word traps you need to look out for.

4.2.1 Words with special meanings: *differentiate, series, number,* etc

Much of the special maths vocabulary consists of normal English words that have been repurposed and given a more-or-less special meaning. Once a certain keyword is used, it starts cropping up in other sentences unbidden—the mind simply latches on and repeats itself. In speech, this is less important; in writing, it looks bad and leads to confusion. For example, in English the verb *to differentiate* means *to recognise the differences between two objects* and the noun *series* means *a number of events or objects coming after one another*. So you could write something like this:

> Our algorithm is powerful enough to differentiate between the power series and the differential of the power series, but it has a series of problems tackling Laurent series, despite the number of improvements we have made since our last paper.

Or something like this where the words are similar without being identical:

> We give a <u>*basic*</u> proof the space X is Hausdorff, by using the <u>*basis*</u> $\mathcal{B}$ defined above.

It helps to be aware of the specialised words[2] in your area of mathematics that may clash with other words in ordinary usage, but ultimately the only way to check for such clashes is to reread your work, both immediately and after enough time has passed that your mind is able to recognise the unwanted repetitions and echoes.

[2]For example, you may think that the name of an ancient Mesopotamian temple, *Ziggurat*, is not a specialised word. It is; see this paper on "Ziggurats and rotation numbers": https://arxiv.org/abs/1110.0080

4.2.2 Relative clauses beginning with *which*, *that* and others

Relative clauses modify a preceding noun (or noun phrase) and typically begin with one of the relative pronouns *which* or *that*, *who* (and the associated forms *whom* and *whose*), or those with adverbial function such as *where* or *when*. Of these pronouns, *which* and *that* are most commonly used in mathematical English, so let us begin with them. Can you explain the difference between these two sentences?

A. The argument *that we explained in the Introduction* works only for Hausdorff spaces.

B. The argument, *which we explained in the Introduction*, works only for Hausdorff spaces.

In Sentence A, the underlined clause specifies which argument the author is referring to; Sentence B assumes the reader already knows which argument is under discussion and the comma-separated, underlined clause reminds the reader the argument was also already explained in the Introduction.

The two sentences exhibit the difference between a *defining* (*restrictive*) and a *non-defining* (*non-restrictive*) clause. A defining clause begins with *that* and removing the clause would result in a loss of meaning. A non-defining clause begins with *which* and is bounded by commas,[3] and it provides additional information. Removing a non-defining clause does not affect the clarity of the main statement.[4] Which of the following is correct?

A. The extreme value theorem applies to continuous, real-valued functions on intervals *that are closed*.

B. The extreme value theorem applies to continuous, real-valued functions on intervals, *which are closed*.

Removing the clause would make the statement false. Therefore, A is correct. In this situation one could have used a simpler phrasing:

> The extreme value theorem applies to continuous, real-valued functions on closed intervals.

In more complicated mathematics, however, you will be forced to juggle *that* and *which* clauses.

[3]A non-defining clause at the end of a sentence begins with a comma and ends with a period.

[4]Technically speaking, from the point of linguistics, a defining clause could also begin with *which*. So sentence A could be written as:

> A'. The argument *which* we explained in the Introduction works only for Hausdorff spaces.

However, this practice is strongly discouraged in mathematical English because of the potential confusion over whether the clause is restrictive or not.

On the other hand, there is no leeway for non-defining clauses. Writing Sentence B as follows is wrong:

> B'. The argument, *that* we explained in the Introduction, works only for Hausdorff spaces.

Let us move on to other relative clauses. The pronouns *who*, *whom*, and the possessive determiner *whose*[5] are used for referring to people in different ways: as those performing an action, as those being subjected to an action, or as those possessing something, respectively. Similarly to before, restrictive clauses cannot be removed, and non-restrictive clauses are separated by commas and can be removed without affecting the meaning. In each of the following pairs of examples, the relative clause in A defines its antecedent, while the relative clause in B merely give more information.

A. The professor who visited the ETH last year gave a good lecture.
This works if there was only one professor who visited.

B. The professor, who visited the ETH last year, gave a good lecture.

A. The professor whom we saw in the hall gave a good lecture.
Of the many professors, the speaker refers to the one seen in the hall.

B. The professor, whom we saw in the hall, gave a good lecture.

A. The professor whose textbook I read gave a good lecture.
This works only if the speaker read a single textbook by a professor.

B. The professor, whose textbook I read, gave a good lecture.

Finally, the relative adverb *where* is often used in mathematical English to tack on a definition that should have been added earlier.

> Suppose the function f is a blue flamingo, where we define a function to be a *blue flamingo* if it satisfies the following conditions...

Whenever possible—and especially if defining something as exotic as a blue flamingo—you are encouraged to give the relevant information prior to using it.

There are exceptions. Consider the following example:

A. Let $O \subset \mathbb{R}^n$ be open and $f\colon O \to \mathbb{R}$ be smooth.

B. Let $f\colon O \subset \mathbb{R}^n \to \mathbb{R}$ be smooth, where O is open.

Here option A is a standard way of defining a function that avoids the "lazy" relative clause. However, in certain cases option B is also acceptable as a way of emphasising the importance of one object over another (namely, in the example, of emphasising the function f over some open set O).

[5] *Whose* can also be used for indicating that an object belongs to a previously mentioned object. E.g.

> The argument, *whose* main purpose was to prove the theorem, can also be used to compute examples.

4.2.3 Commonly confused words

Some words are easily substituted for others by mistake when their letters are transposed or when their meanings are temporarily confused. Worst of all, in those cases the spellchecker may not be able to detect the errors (e.g. when putting *then* instead of *than*, or *chose* instead of *choose*). Luckily, there is a standard set of errors that most of us make and just being aware of it allows for systematic corrections.

Firstly, consider the words in Table 4.4; they sound similar, but are spelled differently and mean different things. Sometimes a slip of the keyboard can make *we choose a point* into a past action *we chose a point*; sometimes the difference is between *the complement of a set* (some other set) and the *the compliment of a set* (flattery received from the set).

Secondly, consider the words in Table 4.5: those in the left column end in *-ic*; those in right end in *-ical*. Unless otherwise specified, the words are adjectives formed from a stem by adding one (or sometimes either) ending. No simple rule exists, so caution is advised in all cases.

- *Algebraic* is a word, but *algebraical* is not (common in maths).
- *Geometric* and *geometrical* are synonyms, but there may be a fixed way of referring to mathematical concepts: you might talk about a *geometric progression* but about *geometrical problems*.
- *Academic* means *relating to scholarship*, but is also a noun meaning a *scholar at a university*. Meanwhile, *academical* denotes matters related to the university in phrases such as *academical year*.[6]
- *Critic* denotes a person (not to be confused with *critique*), while *critical* can mean either *disapproving* or have a specific mathematical meaning, such as in *critical point*.
- *Physic* is a word that the dictionary recognises, so if you were aiming for *physics*, this error will not be caught. On the other hand, *physical*, which means *related to the body* or *relating to the laws of physics*, and is used in various contexts such as *physical attack* or *physical laws*, would not be used in *physical textbook*, where you mean *physics textbook*.
- Sometimes a tired brain can compound the errors, so whilst you mean *topological*, you may write *topographic* (a valid word), which may seem like the right choice as you avoided the errors of *topographical* ("clearly" wrong) and *topologic* (also "clearly" wrong).

The tabulated words form adverbs by adding an *-ally* or *-ly* to *-ic* or *-ical*, respectively. But the rule is not without exception.[7] For example, it is a *public lecture*, but *one lectures publicly*; the adverb is formed by adding a *-ly* to the *-ic* ending of *public*. Whenever you are unsure whether something is a word, check a dictionary, or even better, check to see what textbooks, math papers, or Google say with

[6] If you are Terry Pratchett, you might also consider bringing back the old British word *academicals*, meaning *formal university attire*, and title your book *Unseen Academicals*.

[7] Double negative alert. How would you rewrite this sentence without negative words?

respect to the word or the particular phrase you are writing: the option with the most hits is usually the correct one. Slowly (proofreading) does it.

Finally, here are a few miscellaneous groups of words that might get swapped.

- *Then* vs *Than*: The difference is between an adverb (*then* indicates a sequence or conditional) and a preposition or conjunction (*than* compares and contrasts).
  - Let us write the second proof, *then* compare the two.
    (*sequence*)
  - If the second proof is shorter, *then* we will abandon the first.
    (*conditional*)
  - Actually the first proof is shorter *than* the second.
    (*as preposition in a comparison*)
  - Rather *than* abandon the second proof, let us include both proofs.
    (*as a conjunction in a contrast*)
- *Fewer* vs *Less*: The difference is between the comparative forms of *few* and of *little*. The former is used with countable nouns and people; the latter with mass nouns.
  - This function has *fewer* critical points (than some other function).
  - This project costs *less* to complete (than some other project).
- *Compare with* vs *Compare to*: The difference is generally between carrying out an analysis of properties (comparing apples *with* oranges and deciding they are of the same size) and drawing a parallel or likening (once you have decided the sizes are the same, you can compare the size of the apples *to* that of the oranges). A similar guideline applies to *comparable with/to*.
  - You cannot *compare* your geometric proof *with* my algebraic proof.
    (Because they belong to separate areas of mathematics.)
  - But the complexity of my proof can be *compared to* the complexity of yours.
    (It is of *comparable*, meaning similar, complexity *to* yours.)

  How to remember: *to* is used if you have already carried out an analysis and have concluded the two things are approximately the same. Such conclusions are actually rare in maths (few things are approximate), so chances are that you should use *with*.
- *Consist, comprise, compose, constitute*: If you wish to speak about parts and a whole, you should be well-served with the words *consist* and *comprise*. The most important difference is that in the active voice *consist* takes *of*, whereas *comprise* does not. For example:
  - The proof *consists of* three parts.
  - The proof *comprises* ~~of~~ three parts.

accept (to receive)	except (not including)
adapt (to modify)	adopt (to take on)
advise (to offer suggestions)	advice (*the* suggestion)
affect (usually: to make a change)	effect (usually: *the* change)
alternate (to switch between)	alternative (another choice)
beside (next to)	besides (in addition to)
choose (present tense)	chose (past tense)
compliment (flattering remark)	complement (the rest)
continual (repeated frequently)	continuous (not discrete)
discreet (careful)	discrete (not continuous)
device (equipment)	devise (to invent)
efficient (not wasteful)	effective (successful)
its, their (possessive)	it's, they're (it is, they are)
loose (not fixed)	lose (fail to win or to retain)
practise (to exercise a skill repeatedly)	practice (*the* exercises)
precede (to come before)	proceed (to go ahead)
principal (main)	principle (rule)
stationery (writing materials)	stationary (fixed)
topographical (in geography)	topological (in maths)
whether (consider alternatives)	weather (sunshine or rain)
which (relative pronoun)	witch (not a Muggle)

Table 4.4: Pairs of commonly misspelled or confused words.

-ic	*-ical*
academic (also noun)	academical
algebraic	\
arithmetic (also noun)	arithmetical
\	biological
\	chemical
critic (person)	critical (disapproving, or maths specific)
geometric	geometrical
\	grammatical
linguistic	\
logic (noun)	logical (adjective)
\	mathematical
mechanic (person)	mechanical (operated by machine)
music (noun)	musical (adjective)
physic (arch. drugs)	physical (relating to the body)
public	\
systematic	\
tactic (noun)	tactical (adjective)
topic (noun)	topical (adjective)
\	topological

Table 4.5: Only some words support both *-ic* and *ical*. Pairs of words with different meanings are indicated by bracketed comments.

To compose usually refers to writing a poem or creating a piece of music, but we can say a whole *is composed of* parts. *Comprises* can be fit into the same passive construction. The following sentences are common enough, though occasionally discouraged by style guides:

– The proof *is comprised of* three parts.
– The proof *is composed of* three parts.

Constitute is used in the reverse sense: parts *constitute* a whole. You could say the following, though it may sound a bit strange in maths:

– These three parts *constitute* the proof.

4.2.4 British versus American spelling

Early on in a formal piece of writing you will have to decide whether to adhere to British English or American English spelling and punctuation conventions. We will deal with punctuation differences in the next lecture, but here are a few of the spelling differences.

As these notes use British conventions (with a few exceptions), we put the American conventions second in the examples below. This is merely because an ordering had to be chosen; neither convention is superior to the other, unless the country you are in, the university you are at, or the journal you are writing for has a preference (in which case you honour that preference).

This list is far from exhaustive, both in type and in examples. It is meant to give you an idea what to look out for. Longer lists can be found on Wikipedia and dictionary websites.

pattern	*Brit.*	*Am.*
-ce/-se	practice/practise[8], defence	practice/practice, defense
-ce/-se	advice/advise, device/devise	
-ise/-ize	summarise, emphasise, minimise[9]	summarize, emphasize, minimize
-ise	advise, arise, compromise, exercise, premise, revise, supervise	
-ize	seize, size	
-re/-er	centre, fibre, metre	center, fiber, meter
-er	border, number, quarter	
-re	acre, mediocre, ogre	
-ll/-l	fulfil, enrol, skilful	fulfill, enroll, skillful
-ogue/-og	analogue, catalogue	analog, catalog
-our/-or	behaviour, colour, humour	behavior, color, humor
misc.	maths, orientate, specialism	math, orient, specialty

Table 4.6: British versus American spelling.

[8] Pairs *-ice/ize* as seen in Table 4.4.

[9] By and large the *-ise* ending, especially in verbs, is considered correct in British spelling, where "correct" depends on which style guide you consult. What matters most is being consistent throughout. Though, there are words that always end in *-ise* or *-iz*; see the table for some of them.

Problem Sheet 4

Problem 1. In each case, choose the one option that could replace the underlined phrase and thereby improve the whole sentence.

i) We omit the second computation *owing to the fact that it is* similar to the first.

   A. due to the fact that it is
   B. on account of it being
   C. because it is
   D. whether it is

ii) *On account of* the dynamic nature of the problem, we had to develop a different approach.

   A. The reason is because
   B. Due to
   C. By
   D. Taking into account fact of

iii) We were finally able to prove the theorem *by means of* this method.

   A. on account of
   B. by
   C. because
   D. in

iv) *For the purpose of* this proof, we have developed a specific toolkit.

   A. For
   B. In order that we may tackle
   C. To
   D. Although

v) *A small number of* significant results have appeared in recent years, but none of them have proved the conjecture.

   A. A considerable number
   B. Many
   C. A few
   D. Some

vi) *In the event that* the conjecture is true, then our theorem has the following corollary.

A. Because
B. Consider
C. So
D. If

PROBLEM 2. In each case, choose the one sentence with the correct relative clause.

i) A. The Riemann hypothesis that was named after Bernhard Riemann is celebrated for its difficulty.
B. The Riemann hypothesis which was named after Bernhard Riemann is celebrated for its difficulty.
C. The Riemann hypothesis, which was named after Bernhard Riemann, is celebrated for its difficulty.
D. The Riemann hypothesis, who was named after Bernhard Riemann, is celebrated for its difficulty.

ii) A. Riemann, who wrote his dissertation under Gauss, proposed the famed conjecture in 1859.
B. Riemann, whom wrote his dissertation under Gauss, proposed the famed conjecture in 1859.
C. Riemann, which wrote his dissertation under Gauss, proposed the famed conjecture in 1859.
D. Riemann, who's dissertation was written under Gauss, proposed the famed conjecture in 1859.

PROBLEM 3. Choose the sentences that use the correct underlined word(s).

i) There is one correct solution.

A. We find there are three distinct solutions, rather <u>*than*</u> the expected five.
B. We find there are three distinct solutions, rather <u>*then*</u> the expected five.

ii) There is one correct solution.

A. If we can show there exist more <u>*then*</u> two solutions, <u>*then*</u> the lemma shows there exist infinitely many.
B. If we can show there exist more <u>*than*</u> two solutions, <u>*than*</u> the lemma shows there exist infinitely many.
C. If we can show there exist more <u>*then*</u> two solutions, <u>*than*</u> the lemma shows there exist infinitely many.
D. If we can show there exist more <u>*than*</u> two solutions, <u>*then*</u> the lemma shows there exist infinitely many.

iii) Some sentences may appear to be related; this does not imply that only one of them is correct (or indeed any of them). Consider each sentence individually.

A. The *principal* reason for trying this approach...

B. This is known as Plank's *principal*.

C. Let us now discuss the basic *principle* behind this proof.

D. *Chose* p to be a point in M, such that...

E. We can *choose* either point.

F. To complete the proof, we must *adopt* a different method.

G. To complete the proof, we must *adapt* our previous method to include the new condition.

H. The laws of *physics* cannot be violated.

I. The *physical* laws cannot be violated.

J. The *physical* lectures *that* I attended at university were taught by a Nobel Prize winner.

K. He gave me some *practic advice*.

L. He gave me some *practical advise*.

M. *Witch* function are you referring to?

N. I cannot decide *whether* or not to apply for this position.

O. I will *proceed* with this application because I see no *alternate*.

P. I will *proceed* with this application because I see no *alternative*.

Q. I will *precede* with this application because I see no *alternative*.

R. No argument can *consist* of less than four separate parts, one for each of the four variables.

S. No argument can *comprise* less than four separate parts, one for each of the four variables.

LECTURE 5

Pretty Punctuation

Some would say that spelling errors are the quickest and surest way to discourage a reader before they even reach the mathematical content. But whilst spelling errors suggest the obvious—that you have not proofread your work—punctuation errors can be a more insidious influence, subtly misdirecting the (unaware) reader. The following work by famous American poet ee cummings[1] is an extreme example of how punctuation (and typography) *can* influence the interpretation and overall effect.

here's a little mouse)and
what does he think about, i
wonder as over this
floor(quietly with

bright eyes)drifts(nobody
can tell because
Nobody knows, or why
jerks Here &, here,
gr(oo)ving the room's Silence)this like
a littlest
poem a
(with wee ears and see?
tail frisks)

(gonE)
"mouse",

We are not the same you and

i, since here's a little he
or is
it It
? (or was something we saw in the mirror)?

therefore we'll kiss; for maybe
what was Disappeared
into ourselves
who (look). ,startled

—ee cummings

[1]This is how his name is often stylised

Bear in mind this poem as the antipodal point of what we are aiming for. Let us now turn to the punctuation rules that we wish to *uphold* in mathematical English. We cover all the common punctuation symbols (period, comma, colon, hyphen, quotation marks, question and exclamation marks, parentheses, ellipses) and some of the less-discussed ones (semicolon, en dash, em dash) as they occur in mathematical English. Wherever appropriate we also point out some common errors (mostly related to commas and hyphens). Finally, we also discuss the instances where variations in punctuation consistently occur (e.g. *three-dimensional* or 3-*dimensional*).

5.1 The full stop

The full stop, which is also called the *full point* or *period*, marks the end of a sentence. Merely putting a full stop at the end of a collection of words, however, does not make that collection of words a sentence. For example, this is not a sentence:

> A sentence.

Rather, it is a sentence fragment (allowed in creative writing, but not allowed in formal mathematics). Meanwhile, this is a sentence:

> This is not a sentence.[2]

The same logic applies to in-line and displayed equations. Just because a collection of words and symbols ends in a full stop it is not a sentence:

$$\text{Set } s.$$

The following, however, are two valid sentences:

$$\text{Let } R := \{s \mid s \notin s\}. \text{ Then } R \notin R \iff R \in R.$$

Full stops are also used in numbers, names, abbreviations, and acronyms.

5.1.1 Numbers

Keep in mind that English uses the decimal point ($e \approx 2.718$), not the decimal comma ($e \approx 2,718$) used in German, French, Italian and many other languages.

5.1.2 Names

In bibliographies, the first and middle names of authors are given as initials, followed by full stops, e.g. *B. Mandelbrot*, *A. N. Whitehead*, *L. Lamport*. However, in the body of the text, we usually write either only the last name when referring to illustrious others whose work has already been referenced (e.g. *Whitehead proved...*), or we give the full name as a way of introducing that person to the audience. For instance, in Section 5.1.4 below, we refer to *Leslie Lamport* because we assume most readers are not familiar with him (and we have not included a bibliographic reference that would contain further information).

[2]This is an example of the Grelling–Nelson self-referential paradox. It should remind you of Russel's paradox for sets, which led to the Zermelo–Fraenkel set theory.

5.1.3 Abbreviations

Abbreviations are formed by omitting the ends of words. Some abbreviations are so standard they are hardly thought of as abbreviations at all (log for *logarithm*, tan for *tangent*, and so on). Other non-mathematical abbreviations do not usually appear unless they one of those listed in Table 5.1.

abbrev.	Latin	English
cf.	confer	compare
e.g.	exempli gratia	for example
i.e.	id est	that is
etc.	et cetera	and other things
et al.	et alia	and other (people)
n.b. or N.B.	nota bene	note well
q.v.	quod vide	on this (matter) go see

Table 5.1: Latin abbreviations that occur in maths.

Note that *cf.* should be used to refer to another text of authority, while *q.v.* should be used to refer to another place in the same text. Nevertheless, the latter is rare, and *cf.* is often (ab)used instead of *q.v.*. As discussed for *i.e.* and *e.g.* in Section 2.2.11, in formal work you are almost always better off using the English equivalent of the Latin phrase.

REMARK 5.1. The abbreviation *iff*, whilst handy in informal work, is written out full in a sentence as *if and only if*.

5.1.4 Acronyms and initialisms

Acronyms are formed from the initial letters of a word and usually pronounced as a word themselves. Again, they are rare in mathematical English, but you might get them occasionally, as in the case of the HOMFLY polynomial, where *HOMFLY* is the combination of last-name initials of the mathematicians who discovered it: Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter. You do not put full stops between the letters.

Initialisms are formed in the same way as acronyms, but they are not pronounced as words. So the *Zermelo–Fraenkel set theory* is shortened to *ZFC* (the *C* stands for *choice*) but you pronounce it as a set of letters. Likewise, it is not punctuated with full stops.[3]

Mixed creatures like LaTeX are rare. The collection of symbols LaTeX stands for *Lamport* TeX, where TeX is the typesetting system developed by Donald E. Knuth and Leslie Lamport is the man who modified it to make more a more suitable environment for academics (and mathematicians!).

5.2 The comma

After the full stop, the most important punctuation symbol is the humble and versatile comma. Used judiciously, commas clarify a sentence, give it rhythm, and

[3]There are exceptions, of course, e.g. *U.S.* is sometimes written for United States etc.

help integrate symbolic expressions. Used poorly, commas confuse the reader and cause a lot of trouble (e.g. *Let's eat Mum!*, in which you suggest consuming your own Mum, as compared to the intended, *Let's eat, Mum!*, in which you are calling your Mum to the table).

5.2.1 Restrictive comma

In Section 4.2.2 we explained the differences between the defining or restrictive relative clauses (that take no comma) and non-defining or non-restrictive relative relative clauses (that require a comma).

5.2.2 Serial or Oxford comma

A *serial* comma is a comma that separates off the final element in a list of three or more items joined by *and* or *or*. Its preferred presences or absence is much debated, and some people in general have strong opinions on the subject. Nevertheless, there is no doubt that the serial comma can resolve ambiguity in certain cases. Consider the following example:

> Let us discuss the function f, the shape of its graph and its derivative.

What is to be discussed: the shape of the function's graph and the shape of its derivative, or the shape of its graph and the derivative itself? A patient reader might wonder what is the *shape of the derivative* and from there deduce the correct meaning. To improve clarity instantly, add the comma:

> Let us discuss the function f, the shape of its graph, and its derivative.

Of course, depending on the context and if at all possible, remove any unnecessary words; perhaps you discussing just the *graph* works as well as discussing the *shape* thereof. To shorten, write:

> Let us discuss the function f, its graph, and its derivative.

In this final iteration, the serial comma is not strictly necessary for clarity, though these lecture notes recommend simply always using it because doing so removes the worry that it might be omitted at a crucial location.

5.2.3 Comma separating qualitative adjectives

Qualitative adjectives describe a quality of the noun (e.g. *long*, *good*, *productive*). They have comparative and superlative modes (*longer*, *longest*, *better*, etc.), unlike *classifying* adjectives that do not (e.g. *Swiss*, *black*, *economic*). A comma is required to separate qualitative adjectives, but not other combinations.

Remark 5.2. Note that qualitative adjectives can usually be swapped around, and if needed, you can insert an *and* between them. Classifying adjectives cannot generally be swapped around.

Two qualitative adjectives that need a comma:

- BAD: The subject has a long well-known history.
- GOOD: The subject has a long, well-known history.

A qualitative and a classifying adjective that do not need a comma (and you certainly cannot swap them around):

- BAD: Let m be the smallest, non-negative integer such that...
- GOOD: Let m be the smallest non-negative integer such that...

A qualitative and two classifying adjectives that do not need a comma:

- BAD: There is no long-standing, Swiss economic problem.
- GOOD: There is no long-standing Swiss economic problem.

Two qualitative and two classifying adjectives, where the comma is needed between the first two (but the two classifying adjectives could be swapped around):

- BAD: This is a famous beautiful Swiss historical landmark.
- BAD: This is a famous beautiful Swiss, historical landmark.
- BAD: This is a famous beautiful, Swiss historical landmark.
- GOOD: This is a famous, beautiful Swiss historical landmark.

5.2.4 Comma after introductory elements

Introductory sentence elements such as adverbs, adverbial phrases, and subordinate clauses are often separated with commas.

- The argument is too imprecise. *Nevertheless,* it shows which ideas we need to develop.
- *Before applying this method,* we must check that the function is continuous on the interval.
- *To complete the proof,* we must show the lemma is true.

If the introductory element pertains to time and location, and is sufficiently short, then the comma is optional. Any of the following would do:

- In the Appendix we discuss...
- In the Appendix, we discuss...
- In 2018 we showed that...
- In 2018, we showed that...

You might also consider not putting a comma after *thus*: *Thus we show that....*

But be warned that omitting a comma even after a single introductory word can in some cases change the way it is interpreted:

- Before, the function and its critical submanifolds...
- Before the function and its critical submanifolds...

The first example implies the sentence is referring to something that has been discussed earlier in the same text. The second example implies *before* is the first word of an adverbial phrase that is still not complete in the given fragment. The verb resolves the dilemma, and the following would be two valid sentence continuations:

- Before, the function and its critical submanifolds were easy to compute.
- Before the function and its critical submanifolds can be computed, we must...

Removing the comma after *before* in the first statement or putting it in the second would be incorrect.

5.2.5 Punctuating *however*

A common mistake relates to the use and punctuation surrounding the word *however* (and similarly *therefore*, *nevertheless*, *moreover*). The word *however* is correctly used to introduce a contrasting statement, not to connect two main clauses (this is called a *comma splice*); if you need to connect two main clauses use *but*.

- BAD: The method is flawed, however, there is a way to improve it.
- BAD: The method is flawed, however there is a way to improve it.
- GOOD: The method is flawed; however, there is a way to improve it.
- GOOD: The method is flawed. However, there is a way to improve it.
- GOOD: The method is flawed, but there is a way to improve it.

The word *however* can be used in the middle of the sentence as a way of indicating contrast, and in that case *however* is enclosed by commas. Note that removing the *however* leaves a single complete sentence.

- GOOD: The method is flawed. There is, however, a way to improve it.
- GOOD: As the method is flawed, however, we cannot use it.

There is no other way to punctuate this sentence.

Lastly, *however* can also modify another word, meaning *in whichever way*. Then it is not an introductory element in its own right, though it may be part of one.

- BAD: However, flawed the method, there is always a way to improve it.
- GOOD: However flawed the method, there is always a way to improve it.
- BAD: The method is flawed. Improve it however, you can.
- GOOD: The method is flawed. Improve it however you can.

5.2.6 Comma splice

A comma splice occurs when two main clauses are connected by a comma or adverb instead of by stronger punctuation elements (e.g. a full stop, a semicolon) or by an appropriate conjunction. In the previous section, we saw an example of comma splice featuring *however*. Here are a few more examples.

Short main clauses are no exception:

- BAD: There exists a proof by example, see [TH08].
- OK: There exists a proof by example; see [TH08].
- GOOD: There exists a proof by example [TH08].

Other options for correct (if clumsy) punctuation include a full stop and putting parenthesis around *see [TH08]*.

Do not be fooled by an *-ing* word that looks like it might be modifying something in the first clause.

- BAD: The lemma is false, adding a condition to its statement is needed.
- OK: The lemma is false; adding a condition to its statement is needed.
- GOOD: The lemma is false, unless we add a condition to its statement.
- GOOD: The statement of the lemma requires an additional condition.

Be on the lookout for mixed (invalid) phrase-patterns such as *Let..., then....*

- BAD: Let x be a zero of f', then we can show...
- GOOD: Let x be a zero of f'. Then we can show...

5.2.7 False introductory phrase

Be wary of putting a comma before what may appear to be a introductory phrase but is not. Introductory phrases can be removed without detriment to the grammatical structure of the remaining sentence. In the following example the first underlined phrase is introductory, the second is not and therefore should not be separated by a comma.

- <u>*Regardless of whether the conjecture is true or not,*</u> our theorem says something new about the space X.
- <u>*On whether the conjecture is true or not*</u> depends the proof of my main theorem and ultimately the success of my academic career.

5.3 The Apostrophe

5.3.1 Possessive

The rules for the possessive forms of nouns are clear. However, as mathematicians we also often have to be careful to get the name of an object exactly right, especially if it is called after a person. Not only should the name be correctly spelled, but there are fixed ways of referring to a theorem or formula that include may or may not include a possessive s in the title. For example, it is *the Harnack inequality, Stokes equation, Brownian motion, Fourier Analysis*, but it is *Stirling's formula, Euler's theorem, Dirichlet's unit theorem, Fermat's last theorem.*

5.3.2 Contractions

There should be none in formal mathematical writing. In lectures or talks where the written material is read out, some standard contractions are allowed for the sake of brevity (e.g. *let's*, *it's*, *that's*) .

5.3.3 Plurals

The plural forms of symbols that are italicised or that otherwise stand out can be indicated with an additional s, such as in:

> There are seventeen Ds.

However in some cases that might look confusing, e.g. αs and xs and 0s, in which case you could use an apostrophe to indicate the plurals:

> I do not deal with α's and x's at work; I only deal with 0's and 1's.

5.4 The colon and semicolon

A colon denotes the start of a list, or the start of a conclusion or emphatic statement. It behaves similarly to phrases such as *that is, namely, because, it follows, for example.* The sentence fragment following a colon does not have to contain a verb or be able to stand alone.

- At university I focused on five subject: statistics, algebra, and programming in the first two years, then thermodynamics and quantum mechanics in my final year.
- I moved to Switzerland to complete my studies: it was worth it!

In contrast, the semicolon occurs between two stand-alone sentences, indicating a separation that is stronger than a comma and weaker than a full stop.

- At university I studied a whole range of subjects; I regret not having more focus.

Colons and semicolons combine effectively to punctuate complex lists, where the elements of the lists themselves contain commas or other punctuation.

- At university I studied a whole range of subjects: calculus, statistics, and abstract algebra in my first year (I also took a Swiss history class!); magnetism, optics, and differential geometry in my second year; and quantum mechanics in my final year, while participating in a seminar on quantum computing.

5.5 The hyphen, the en dash, and the em dash

Figure 5.1: The hyphen, the en dash, the em dash, and the minus sign, respectively, enlarged from the LaTeX typesetting in these lecture notes.

- The *hyphen* is used to join words to form compounds.[4]
- The *en dash* (or *en rule*) is longer than a hyphen and is used to specify ranges of elements or connections between words (similarly to *and*).
- The *em dash* is twice as long as the en dash and is a versatile punctuation sign most often used to highlight a phrase that would otherwise be lost in the middle of a sentence (a parenthetical statement).
- The *minus sign* is longer than an en dash, shorter than an em dash, and at least with the current typographic settings of these notes, neither level with the hyphen nor with the dashes (the minus sign also appears to have rounder corners in this font!).

In the following sections we discuss the first three items in detail.

5.5.1 Hyphenation

A compound word is made up of at least two other words and it takes one of the following forms:

- *Open compound*: the constituent words continue to be written separately, as in *rational number*, *equivalence classes*, *Galois group*, *axiom of choice*, *power series*, *metric space*. Because open compounds are written as separate words, they can be hard to identify. Indeed, only the combinations of separate words

[4]This word-joining hyphen is called the *hard hyphen*. The *soft hyphen* indicates word division at the end of a typeset line and it occurs often in a text that is automatically *justified* (meaning both left and right-hand sides are aligned to a specified margin). You will see soft hyphens a lot when you start writing in LaTeX, and indeed there are many in these lecture notes.

that occur sufficiently frequently are considered compounds. So if your name is Sam and you define *Sam's series*, it is not a compound, unless it becomes well known.

- *Closed compound*: the constituent words are written as one word, as in *supersolvable*, *hydrodynamics*, *hyperelliptic*, *eigenvalue*, *pullback*, *endpoints.*
- *Hyphenated compounds*: the constituent words are combined into a new word using a hyphen or hyphens, as in *well-defined*, *real-valued*, *space-filling*, *second-countable*, *self-contained*, *one-to-one*, *skew-symmetric*, *nowhere-differentiable.*

In everyday English, if you are told a compound word, but have not seen it written down, it may be difficult to guess which of the three forms it takes. For example, it is *living room*, but it is *bedroom*.[5] So it is perfectly normal to be unsure about spelling a compound.

There is, however, a set of guidelines that governs the hyphenation of compounds that modify nouns. They all apply to mathematical English as well—or at least they should, with few exceptions. Here are some of those guidelines.

i. Usually, a compound that modifies a noun is hyphenated if it appears before the noun, and not hyphenated if it appears after the noun.

   - He is a well-known professor.
   - The professor is well known.
   - We have shown that f is a well-defined function.
   - The function f is well defined.
   - This is an up-to-date study.
   - This study is up to date.

ii. A compound containing an adverb that ends with *-ly* is not hyphenated regardless of its position. (This is an exception to i.)

   - They applied a newly discovered method.
   - I work with locally finite groups.
   - The group is finitely generated.

iii. A compound made up of a noun or adjective and verb participle is hyphenated regardless of its position. (This too is an exception to i.)

   - The proof is self-contained.
   - The function is real-valued.

[5] Actually, the situation is more complicated than that. About ten years ago a spelling reform dealt with hyphenated compounds, making some of them into open compounds and some of the others into closed compounds (while the rest remained hyphenated). So you may have learned in school that it was *ice-cream*, except the correct spelling today is *ice cream.* You may have been told that you were (not) a *cry-baby*, except that word is now spelled as *crybaby.* But even that depends which dictionary you check.

Aside from the usual compounds, mathematical English makes use of specialised symbol-and-word combinations.[6] After all we work with *the ϵ-neighbourhood, the x-axis, the p-adic numbers, the n-holed torus, the L^p-norm,* etc. These are undoubtedly always hyphenated.[7] So far so clear: ordinary English (OE) provides the guidelines for the verbal side; mathematical English (ME) provides its specialised symbolic parts. However, there are some differences in practice when it comes to the boundary between the two. Consider the following example:

- We have shown V is an infinite-dimensional space.
  (*OE and ME agree on hyphen.*)

- We have shown that V is infinite dimensional.
  (*OE would say this is correct: no hyphen. ME accepts this in practice.*)

- We have shown that V is infinite-dimensional.
  (*ME also accepts the hyphen in practice.*)

The discrepancy occurs because the guidelines surrounding mathematical English are not precisely fixed; it is not a language *per se*, nor is it a dialect, and it is not usually taught,[8] which means authors choose their own conventions.

To illustrate the fact, let us take a common example: everyone knows that we say something is *n-dimensional*, but what if n is concrete and small? Is it *one-dimensional* or *1-dimensional*? The discussion in Lecture 1 would suggest that writing the numeral is correct, but practice shows otherwise. In fact, searching http://arxiv.org for the different pairs of strings shows a predominance of the former (pure word compounds) over the latter (symbol-and-word compounds) until $n \geq 7$; see Figure ??.

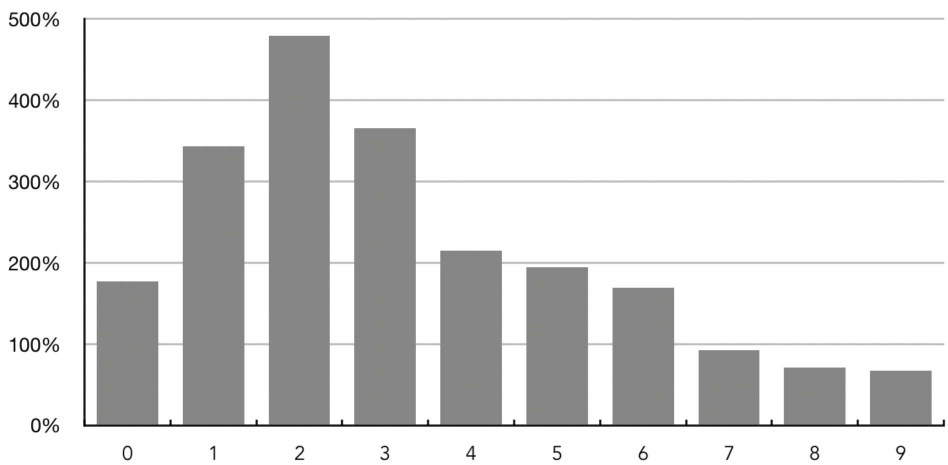

Figure 5.2: Ratio of *word-dimensional* appearances versus *digit-dimensional*.

[6] This is the kind of compound you are free to make up in English to compress longer phrases, as long as the compression is sensible. Here we could have written: *combinations of symbols and words.*

[7] Some publishers like to use an en dash here instead of a hyphen.

[8] Which is why this course is quite unusual!

If further illustration of variety is needed, here is an excerpt from page 5 of the fairly standard text *Riemannian Geometry and Geometric Analysis* by Jurğen Jost. The ellipsis in the quote denote text that was skipped for brevity.

> The only one dimensional manifolds are the real line and the unit circle S^1, the latter being the only compact *one*. Two dimensional compact manifolds are classified by their genus and orientability character. In three dimensions, there exists a program by Thurston[243,244] about the possible classification of compact three-dimensional manifolds. ... In dimension 4, the understanding of differentiable structures owes important progress to the work of Donaldson. He defined invariants of a differentiable 4-manifold M from the space of selfdual connections on principal bundles over it.

The general advice still stands: try to be consistent. If you ever write a textbook yourself, you will be within your rights to punctuate the *dimension* words any way you like (within reason).

5.5.2 The en dash

The en dash closes up ranges of elements or dates.

- Read pp. 705–708.
- We are particularly indebted to Bernhard Riemann (1826–1866) for...

Moreover, the en dash is used between two or more authors to indicate that their joint creation is not the work of someone with a double-barrelled surname, such as in the *Smith–Volterra–Cantor set, the Riemann–Lebesque lemma*, and *Calabi–Yau manifolds.* Not all publishers follow this rule, and indeed a quick flip through the nearest books suggests that AMS and Prentice Hall publications use the hyphen, while Springer and Princeton University Press use the en dash. Wikipedia too uses the en dash in a surprising number of places, though other online sources may not.

5.5.3 The em dash

An em dash can be used to introduce an phrase at the end of the sentence, the way a colon might, and it emphasises that phrase. This is more common in expository sections and informal writing, such as these lecture notes. For example, below we say:

> Since the goal of mathematical writing is precision and clarity, you should always aim for manageable sentences—manageable to write and manageable to read.

But the em dash can also be used effectively when writing application essays.

> Before settling on Hodge theory, I changed the subject of my thesis three times over the course of my Masters—I was determined to find a problem that suited my strengths best.

A pair of em dashes can be used to enclose a phrase, the way you would with a pair of commas or parentheses. The enclosed phrase is called a *paranthetical statement*, and how much attention the reader pays it depends on what punctuation you choose. Consider the following five examples. The only difference is the punctuation.

i. In this section we discuss the counterexamples[9] and how their properties can point the way to a more comprehensive theory.

ii. In this section we discuss the counterexamples (infinitely many nondiffeomorphic manifolds that violate all three conditions of the statement) and how their properties can point the way to a more comprehensive theory.

iii. In this section we discuss the counterexamples, infinitely many nondiffeomorphic manifolds that violate all three conditions of the statement, and how their properties can point the way to a more comprehensive theory.

iv. In this section we discuss the counterexamples—infinitely many nondiffeomorphic manifolds that violate all three conditions of the statement—and how their properties can point the way to a more comprehensive theory.

v. In this section we discuss the counterexamples—infinitely many nondiffeomorphic manifolds that violate all three conditions of the statement!—and how their properties can point the way to a more comprehensive theory.

The difference in effect is as follows:

i. **Footnotes:** Putting a statement in a footnote almost guarantees that the statement will not be seen by most people when they first come across it. Strictly speaking, this is not an example of a parenthetical statement, but it is frequent in technical writing, so we include it for completeness.

ii. **Parentheses**: Parentheses push the parenthetical statement into the background; they indicate that the statement can be glanced over or skipped. Unlike with a footnote, readers will at least register the enclosed content.

iii. **Commas:** Commas ensure that the statement will be given due notice as an equal part of the sentence. However, it does make a sentence more cumbersome especially if the parenthetical statement is long. Commas are a good choice for shorter explanatory phrases (e.g. *The two relevant theorems mentioned above, <u>Theorem 5 and Theorem 6</u>, can be used...*); alternatively rephrase a longer parenthetical statement to make it a new sentence.

iv. **Em dashes**: A statement is emphasised when surrounded by em dashes. They guarantee the reader will pay attention to what is being said.

v. **Em dashes with exclamation mark**: If the parenthetical statement is not only important but also surprising, you may want to include an exclamation mark to highlight this surprise. It should be used sparingly, and only when you are sure that the fact is indeed worthy of such emotion.

[9] Infinitely many nondiffeomorphic manifolds that violate all three conditions of the statement.

If you think of a sentence punctuated by periods and commas as being level ground, then a footnote is an abyss, parenthesis are a valley, commas are level, and em dashes are the hills.

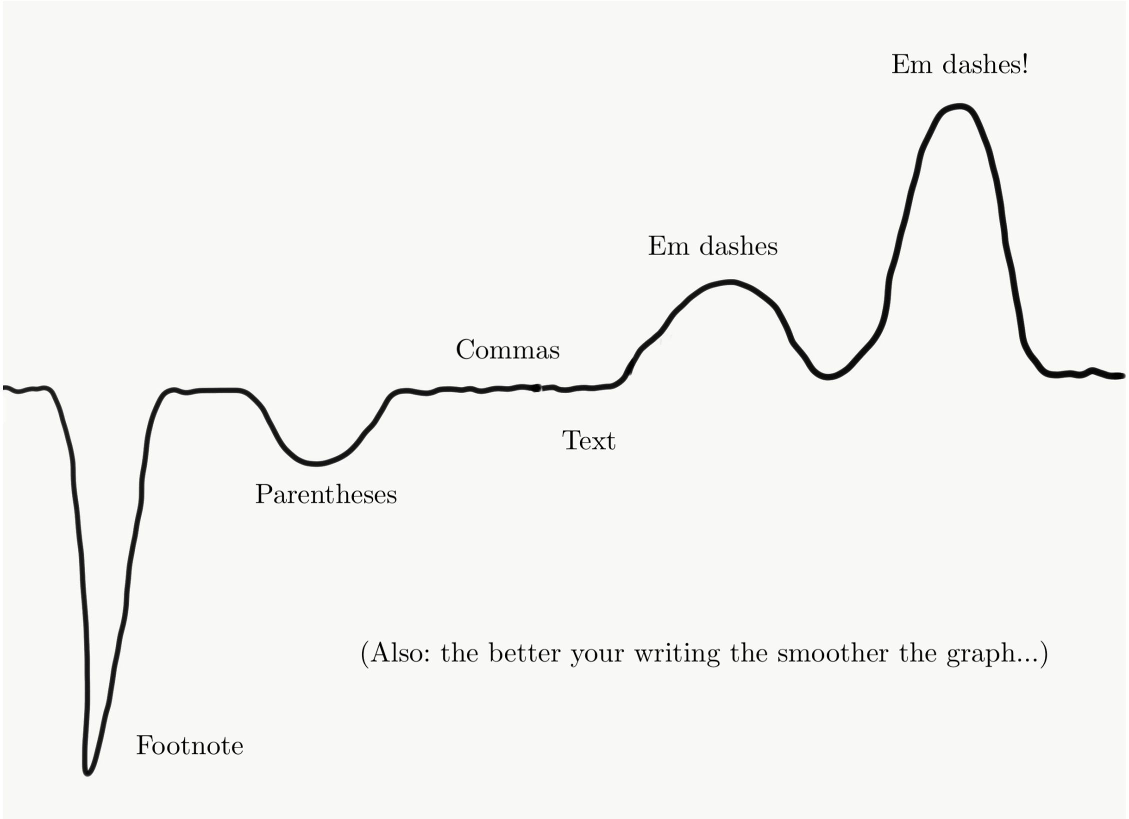

Figure 5.3

Finally, we remark on the punctuation surrounding an em dash. No punctuation should precede a single dash or the first dash of a pair, and the word coming after the dash should not be capitalised unless it is a proper noun. The closing dash of a pair may be preceded by an exclamation or question mark. Whether or not you choose to surround each em dash with spaces is a matter of convention. These notes choose not to do so.

5.6 Quotation marks

Quotation marks are rarely used in mathematical writing outside of bibliographies. Here are three instances in which you may find them helpful.

- **Direct quotation:** You may find yourself having to directly quote a non-standard phrase. For example:

  > We now proceed to discuss the "blue flamingos" that were introduced in the previous lecture.

- **Informal speech:** You may find yourself having to explain a concept in less formal terms, in which case you could reach for an appropriate metaphor or phrase from ordinary English. For example:

> The torus can be foliated in such a way that the leaves resemble "chairs" and the foliation itself can be thought of as a "stack of chairs".

This example contains two words common in English: *leaves*, which is a specialised mathematical term, and *chairs*, which is not (in this context). This crucial difference is shown by the presence of quotation marks.

- **Sarcasm, irony, emotion**: You may find yourself wanting to use an opinion word in a way that signals to the reader you are aware this is an opinion word that may not be justified, but you wish to use it anyway. For example:

  > We proceed to offer a more "elegant" proof.

  The meaning of such statements can only be determined from context.

Unless you are writing a historical study, the use of quotation marks should be kept to a minimum.

5.6.1 British versus American punctuation

The main differences occur with respect to quotation marks. Modern British practice encloses quoted material in single quotation marks '', unlike the American double quotation marks "". Furthermore, the difference extends to where the adjacent punctuation is placed: inside the quotation marks or outside.

- The leaves of the foliation resemble 'chairs', meaning they can... *(British)*
- The leaves of the foliation resemble "chairs," meaning they can... *(American)*

You may have noticed that these notes consistently use the double quotations marks. Indeed, but we use British punctuation and spelling with other few exceptions.

5.7 The question mark and the exclamation mark

In general you should have little use for either of these punctuation marks. Exclamation marks may crop up sometimes in examples such as shown in Section 5.5.3. Questions marks may appear in rhetorical questions that introduce a subject or a historical discussion.

5.8 Brackets and Ellipses

There are at least five types of brackets.

- **Parentheses** or **round brackets** or (): the most versatile kind used both in symbolic expressions and writing.
- **Square brackets** or []: they occur rarely outside of symbolic expressions.
- **Braces** or **curly brackets** or {}: likewise.

- **Angle brackets** or <> (not used in mathematics): they never occur in mathematical English because they are written using *less than* and *greater than* symbols, which have their own specialised meaning.
- **Narrow angle brackets** or $\langle\rangle$: they never occur outside of the symbolic expressions.

5.8.1 Parentheses

As mentioned above in Section 5.5.3, parentheses can be used to enclose an expression that could be useful to the sentence, but is not crucial to it. These notes often put examples or additional information in parentheses, as do many papers and books. Strictly speaking, there is no limit on the length of the parentheses. Usually, the enclosed text is a phrase or sentence fragment, but very occasionally you might want to enclose a whole sentence.

- The functions can be defined in numerous ways, but we shall use the construction described in [HV54] (the other methods are beyond the scope of this paper).
- The functions can be defined in numerous ways, but we shall use the construction described in [HV54]. (The other methods are beyond the scope of this paper, though we intend to employ [HV55] in a future paper.)

In the first example, the parenthetical expression did not warrant a separate sentence, and the final full stop appears *after* the parenthesis.

In the second example, the parenthetical expression warranted a separate sentence because of the added information, which lengthened it and gave it additional weight. Here the final full stop or any end punctuation pertaining to the statement would occur *before* the final parenthesis.

Nested parentheses outside of symbolic expressions are rare.

5.8.2 Ellipses

Ellipses have specific uses in symbolic expressions and do not appear otherwise.

5.9 Capitalisation

The rules differ, especially when it comes to titles. Certainly the first letter of any title should be capitalised, but after that you should be governed by the publisher's in-house style or simply by common sense. As always, what matters most is consistency.

For these lectures notes, we choose to capitalise all the words in the main title that are not prepositions or articles—a common practice for book titles, for example—but we choose not to capitalise anything other than the initial letter (excepting proper nouns) in all the subsection and subsubsection headings. We did this to reflect a more informal style.

Mathematical objects are capitalised according to whether or not they are named after someone, though this is something you have to learn along the way.

- Is it *de Rham cohomology* or *De Rham cohomology*? (The former, unless it begins a sentence.)
- Is it *Abelian group* or *abelian group*? (Apparently there is a tendency towards the latter, but the former is not incorrect.)
- Is it *Kroneker delta*, *Kroneker's delta*, *Kroneker-delta* or *kroneker delta*? (The first option, though the second and third options appear on arxiv.org too.)

If in doubt, check a number of established sources (textbooks, well-known papers) before deciding for yourself how to capitalise and punctuate a word.

Areas of mathematics are capitalised when they are treated as a proper name of a subject.

- We proceed to define a new homology theory.
  (*homology* modifies the common noun *theory*)
- I have decided to study Homology Theory.
  (*Homology Theory* is a subject in its own right.)

5.10 Punctuation "forests"

Since the goal of mathematical writing is precision and clarity, you should always aim for manageable sentences—manageable to write and manageable to read. This means keeping the sentences shorter rather than longer and the word choice simpler rather than more complicated. It also means avoiding punctuation "forests", where numerous punctuation marks jostle for attention, perhaps together with italics, bold, a few acronyms, and some notation.

The following is a grammatically correct sentence—a single one!—punctuated according to the rules. In theory, its meaning is unambiguous. In practice, the mathematical content is garbage, but to determine it is garbage you first have to wade through a visually and linguistically onerous block of text. Most people would give up. Most people *will* give up even if the mathematics turns out to be correct, so try to avoid anything remotely as dense as this.[10]

> As we will later show in detail, this is, however, not a method that always works for any vector space V and its associated projective special linear group $PSL(V)$ (e.g., see [YHS78]; they give *eight* counterexamples!); indeed, of all the the Mathieu groups—the five sporadic simple groups M_{11}, M_{12}, M_{22}, M_{23}, and M_{24} that Mathieu introduced in the second half of the nineteenth century—already M_{12} has a maximal subgroup of order 660, so how could this method work on a group as complex as the largest sporadic simple group, **the Monster group**, of order 808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000?

[10] Tackling a poem by ee cummings seems a comparatively easy task now, does it not?

Problem Sheet 5

Problem 1. Choose the best-punctuated option (where the punctuation aims to clarify the meaning).

A. These lecture notes are designed to help you eliminate unnecessary words, causes of confusion and common mistakes from your writing.

B. These lecture notes are designed to help you eliminate unnecessary words, causes of confusion, and common mistakes from your writing.

C. These lecture notes are designed to help you eliminate unnecessary words and causes of confusion, and common mistakes from your writing.

D. These lecture notes are designed to help you eliminate causes of confusion and common mistakes, and unnecessary words from your writing.

Problem 2. Choose the option that is correctly punctuated.

A. Their theorem uses a geometric argument. Nevertheless, the same result can be proved using the argument we outlined.

B. Their theorem uses a geometric argument, nevertheless, the same result can be proved using the argument we outlined.

C. Their theorem uses a geometric argument, nevertheless the same result can be proved using the argument we outlined.

D. Their theorem uses a geometric argument nevertheless the same result can be proved using the argument we outlined.

Problem 3. Choose the option that is correctly punctuated.

A. However, you choose to rephrase the problem, the underlying difficulty remains.

B. However you choose to rephrase the problem the underlying difficulty remains.

C. However, you choose to rephrase the problem, the underlying difficulty remains.

D. However you choose to rephrase the problem, the underlying difficulty remains.

Problem 4. Choose the option that is correctly punctuated.

A. Each textbook offers a different proof, I prefer the one given in [Ha00].

B. Each textbook offers a different proof, the one given in [Ha00] is preferable.

C. Each textbook offers a different proof, however I discuss the one in [Ha00].

D. Each textbook offers a different proof; we discuss the one given in [Ha00].

Problem 5. Choose the option that is correctly punctuated.

A. Let G be a subgroup of H satisfying the two conditions, then suppose G has order n and proceed by induction.

B. Let G be a subgroup of H satisfying the two conditions, suppose G has order n and proceed by induction.

C. Let G be a subgroup of H satisfying the two conditions. Suppose G has order n and proceed by induction.

D. Let G be a subgroup of H satisfying the two conditions, then supposing G has order n, we proceed by induction.

Problem 6. Choose the <u>two</u> options that are correctly punctuated.

A. Developing the new homology theory, will take more time than I have during my PhD studies.

B. Developing the new homology theory will take more time than I have during my PhD studies.

C. To develop the new homology theory, I will need help from my adviser.

D. To develop the new homology theory I will need help from my adviser.

Problem 7. Choose the correctly punctuated list.

A. The book on 4-manifolds is split into three parts—the first part covers definitions, examples, surfaces in 4-manifolds, and the blow-up process—the second part covers handle decompositions, Heegaard splittings, Kirby diagrams, surgery, and Spin structures—and finally, the third part covers elliptic and Lefschetz fibrations, cobordisms, symplectic 4-manifolds, and Stein surfaces.

B. The book on 4-manifolds is split into three parts: the first part covers definitions, examples, surfaces in 4-manifolds, and the blow-up process; the second part covers handle decompositions, Heegaard splittings, Kirby diagrams, surgery, and Spin structures; and finally, the third part covers elliptic and Lefschetz fibrations, cobordisms, symplectic 4-manifolds, and Stein surfaces.

C. The book on 4-manifolds is split into three parts: the introduction covers definitions, examples, surfaces in 4-manifolds, and the blow-up process. The second part covers handle decompositions, Heegaard splittings, Kirby diagrams, surgery, and Spin structures. Finally, the third part covers elliptic and Lefschetz fibrations, cobordisms, symplectic 4-manifolds, and Stein surfaces.

D. The book on 4-manifolds is split into three parts: the introduction covers definitions, examples, surfaces in 4-manifolds, and the blow-up process, the second part covers handle decompositions, Heegaard splittings, Kirby diagrams, surgery, and Spin structures, and finally, the third part covers elliptic and Lefschetz fibrations, cobordisms, symplectic 4-manifolds, and Stein surfaces.

Problem 8. Choose all the options with the correct punctuation related to en dashes and hyphens.

A. Options I–J in this problem pertain to an imaginary bibliographic entry.

B. Options I-J in this problem pertain to an imaginary bibliographic entry.

C. We have shown that G is a finitely generated Abelian group.

D. We have shown that G is a finitely-generated Abelian group.

E. Show that f is a one to one function between the two sets.

F. Show that f is a one-to-one function between the two sets.

G. The set S has two elements.

H. We will return to the two-element set S.

I. J. Kervor and P. Meliwawa–Kasan, *The signature of minimally perturbed students*, Communication in Mathematics for ETH students **1** (2018), 456-478.

J. J. Kervor and P. Meliwawa–Kasan, *The signature of maximally pleased students*, Communication in Mathematics for ETH students **1** (2018), 456–478.

K. J. Kervor and P. Meliwawa-Kasan, *The signature of students taking the course*, Communication in Mathematics for ETH students **1** (2018), 456–478.

L. Do you know the statement of the Riemann-Roch theorem?

M. Do you know the statement of the Riemann–Roch theorem?

LECTURE 6

Grammar Gaffes

> You don't know about me without you have read a book by the name of *The Adventures of Tom Sawyer*; but that ain't no matter.
>
> — Mark Twain, *Adventures of Huckleberry Finn*

In everyday and literary English, grammar gaffes are embarrassing and occasionally misleading. In maths writing—assuming you already speak English quite well—it is less a matter of blushes and missed connections, as it is a matter of getting the form right.

We have mentioned before how a creative vocabulary is a hindrance in maths, rather than a boon, and how statements should be put as plainly and clearly as possible, rather than with a flourish. We have talked about conventional phrase-patterns and punctuation. The language elements we have least touched upon are verbs, and they are the ones that allow us to manipulate mathematical objects and express their relationships.

In Section 6.1, we discuss the tense, mood, and voice of verbs in formal mathematical writing. In particular, we usually write using a simple tense, in the imperative or interrogative mood, and in a balanced mixture of the active and passive voice. In Section 6.2, we warn against the temptation to abuse the grammar rules.

6.1 Verbs

In this section we review the relevant features of verbs, offering guidelines on how to refer to parts of the same paper (Is it: *the theorem above* <u>*showed*</u> or *the theorem above* <u>*shows*</u>?) and when to use the passive versus the active voice. We also discuss the common reduction of adjectival clauses to adjectival phrases, known to you in practice as the omission of the words *that* and *that is/are*.

6.1.1 Tense

Table 6.1 gives a brief recap of the verb forms.

The continuous and prefect continuous forms. In formal mathematical writing we have almost no use for the continuous forms. The proofs, the work, the thinking *have been completed*[1]; in other words, they are over and no one cares what you did while you were working. Likewise, any references that you make to past work or even future work, should consider the results of that work, not the process. The way to think about it is this: any mathematical text is built on assumptions

[1]These lectures notes are exempt from tense considerations because they write about writing—an activity that requires more linguistic latitude.

we	Present	Past	Future
Simple	*prove*	*proved*	*will prove*
Continuous	*are proving*	*were proving*	*will be proving*
Perfect	*have proved*	*had proved*	*will have proved*
Perfect cont.	*have been proving*	*had been proving*	*will have been proving*

Table 6.1: The three verb tenses (present, past, and future) and the four aspects (simple, continuous, perfect, and perfect continuous).

about what is known; those assumptions are timeless and "set in stone" as far as the text is concerned.

The exceptions are sections on history or brief insights into the moments of a research project that are interesting in their own right, though only outstanding results in any field will merit either.

The perfect forms. The future perfect (*will have proved*) refers to a an action that will be completed by a certain time, and therefore clashes with the set-in-stone notion. The past perfect (*had proved*) refers to an action that has already been completed before another action happened. Dates and references tell mathematicians when something was done more precisely than verb forms, so the past perfect usually only occurs in historical exposition.

Finally, the present perfect (*we have proved*) can be confused with the simple past (*we proved*). The difference is as follows.

The simple past refers to a concrete point in the past:

- In Theorem 5.4 we proved that...
- We proved above that...
- We proved that...

The last example would occur immediately after a proof, or during a discussion that has first pinpointed the proof.

The preset perfect is more versatile in ordinary English, but in mathematical writing it is limited to emphasis without a specific reference and usually as part of a comparison or contrast.

- They <u>have</u> already <u>proved</u> a version of this result, and therefore we are not the first to do so.
- We <u>have</u> not <u>proved</u> that f is smooth, but we <u>have proved</u> that f is continuous.

The present perfect is the only perfect that you might expect to see in the body of a paper, and even then rarely.

The verb *to do*. The auxiliary verb *to do* can be used similarly for emphasis or to pose a question:

- The continuity argument <u>*does*</u> (indeed) complete the proof.

- *Does* the continuity argument complete the proof?

However, like with the present perfect, this form should be reserved for exceptional cases. For negation with *do*, see below.

The simple forms. Convention dictates using the present tense when discussing current material, so it is by far the most prevalent in a text. However, you have a few options for handling references to other people's work, and more importantly to previous and upcoming parts of your text.

Other people's work is usually referred to in the simple present or the simple past.

- Gabai *proves* the 4-dimensional Light Bulb Lemma in [Ga18].
- In his seminal work on integration, Lebesgue *proved* that...

Both options could work well, though not in the same text. Once you pick a tense, stick to it.

For referencing your own work you have the following three options:

1. Use the present tense for everything.
   - NOW: We *prove* the result by induction.
   - FUTURE: Below, we *prove* the result by induction.
   - PAST: The discussion above *proves* the result by induction.

   Indicating the future using the present plus a reference (e.g. *below*, *later*, *In Section 6*) sounds natural in English. Indicating the past this way is a little harder on the ear. It can be done, for example as shown, by making the subject a non-human entity and therefore rendering the statement closer to being timeless.

2. (**Recommended.**) Use the present tense for present and future references, and use the past for past references. This is an elegant and common choice.
   - NOW: We *prove* the result by induction. (Same as in 1.)
   - FUTURE: Below, we *prove* the result by induction. (Same as in 1.)
   - PAST: Above, we *proved* the result by induction.

3. Use the tenses according to their labels: present for present, past for past, future for future. This is a straightforward, if slightly less elegant choice.
   - NOW: We *prove* the result by induction. (Same as in 1. and 2.)
   - FUTURE: Below, we *will prove* the result by induction.
   - PAST: Above, we *proved* the result by induction. (Same as in 2.)

Regardless of which option you choose, in a summary section, it is common to use the simple past; to talk about your own future work, it is common to use the simple future.

> Most mathematical statements are best expressed using a simple tense.

Negation. If the verb form contains an auxiliary verb then *not* goes after it:

- We *have not proved* the theorem.
- We *will not prove* the theorem.

If there is no auxiliary verb the appropriate form of the auxiliary verb *to do* is added, together with *not*.

- We *do not* prove the theorem.
- We *did not* prove the theorem.

6.1.2 Mood

Grammatical mood indicates the intentions of the speaker. Most of mathematics is written in one of the following two moods.

- The **indicative** mood expresses facts.

  The group G is abelian.

- The **imperative** mood issues instructions.

  Let G be an abelian group.

The other moods occur rarely in formal maths writing. They are:

- The **interrogative** mood asks questions. You would find this mood on a problem sheet or very occasionally as a rhetorical question that introduces a major problem at the beginning of a text.

  Is the group G abelian?

- The **conditional** mood refers to uncertain situations. In a less formal style or conjectural situation, this mood could be used to express expectations of a result coupled with an inability to investigate it further (and a good reason as to why this is the case).

  The group G should be abelian, but it is not easy to show this.

- The **subjunctive** mood expresses a wish or possibility. This mood would only appear in a discussion that explored alternative proofs or options (with some particularly worthy goal in mind).

  If the group G were abelian, then the result would have been true.
  (*But given that G is not abelian, the result may not be true.*)

The subjunctive mood in this example should be distinguished from the typical *if. . . , then. . .* pattern that uses the present tense indicative to state a fact.

> If the group G is abelian, then the result is true.
> (*We do not have an opinion about whether G is abelian.*)

Note that these last three moods convey a degree of uncertainty. Therefore, they are uncommon in a discipline that centres on proving universal truths. If you find yourself writing anything other than the indicative or imperative mood, take a look at your mathematics: your understanding of the theory may be shaky.

> Most mathematical statements are in the indicative or imperative mood.

6.1.3 Voice

The *active* or *passive voice* of a verb describes the verb's relationship with its subject and object. Indeed, words that are the subject and object of an active verb switch roles when the verb is made passive. Traditionally, schools teach that the passive should be avoided unless absolutely necessary. As evidence, minimal examples of active sentences are converted to the passive, yielding statements that are obviously "silly":

- ACTIVE: We *prove* the theorem.
- PASSIVE: The theorem *is proved* (by us).

Alternatively, university instructors warn against the passive because it leads to tedious and "legalese" sentences:

- ACTIVE: We *will prove* the theorem in Section 5 and *use* it to construct examples that support the conjecture we *made* in [RT67].
- PASSIVE: The theorem *will be proved* in Section 5 and it *will be used* to construct counterexamples to support the conjecture that *was made* by us in [RT67].

Silly or legalese, the passive voice is taught as abstract and inferior. As a result, you might think that you should always simply use the active voice. But then you are also told mathematicians work with universal truths that are independent of individual humans, and therefore the *we* or the *I* should not be prominent.[2] For once, the advice is not to be blindly consistent.

> Good writing strikes a balance between a self-centred and an abstract tone.

Let us discuss how to find this balance.

[2]We will discuss whether it is *we, I, you, one, they* in the next lecture.

Understand the middle-ground. The case for silliness and legalese becomes less clear if the sentences are neither curt nor wordy, and especially if the subject is non-human.

- ACTIVE: The continuity argument *completes* the proof.
- PASSIVE: The proof *is completed* by the continuity argument.

Switching around *argument* and *proof* changes the emphasis: in the first case, the reader focuses on what does the action (*the argument)*; in the second, on what is bearing the action (*the proof*). Either is acceptable. Also, in this example, it is even acceptable to reinsert the author as the subject, though an additional word is needed to make the sentence grammatical:

- ACTIVE: We *complete* the proof by *using* the continuity argument.

Understand your options. When writing a maths sentence there are essentially four options:

1. The active voice, with yourself as the subject.
2. The active voice, with a maths term as the subject.
3. The passive voice, with yourself as the object.
4. The passive voice, with a maths term as the object.

Not all the options will be available for every verb. For example, *intransitive* verbs have no object, and therefore they have no passive voice. Notably, these are the verbs *to be* and *to exist*. Likewise, you cannot always shoehorn yourself in as the subject: if a group *acts* on a set, then you cannot say: We *make* the group act on the set, or something similar.

Understand the subtleties. In the end, most people write whatever voice first comes to mind, echoing papers and books they have studied from, and if they err, they usually err on the side of too much passive. However, occasionally even if you realise that something is not quite right, it is hard to identify the issues. Let us work through an example that exhibits some of the characteristic issues.

Consider the following paragraph taken from a textbook and modified to have a mixture of active (yellow) and passive verbs (blue). We have inserted the superscript numbers for ease of reference.

> It is[1] the primary aim of manifold theory to classify topological manifolds, i.e., to give a complete list of n-dimensional (closed) topological manifolds, and to find a way to tell which topological manifolds carry smooth structures (have C^∞-atlases). Furthermore, if one such atlas is found[2], it would be good[3] to determine the total number of these up to diffeomorphism. In most dimensions this aim cannot be achieved[4] for algebraic reasons (cf. Theorem 1.2.33 and Exercise 5.1.10(c)); in those cases further conditions (like simple connectivity) will be imposed[5] for

> the manifolds at hand. For a better understanding of results concerning 4-manifolds, this section will be concluded[6] with theorems concerning manifolds of dimension different from 4. The manifolds we are working[7] with are assumed[8] to be closed, connected and oriented.

Here are few observations:

- The verb in 1 is active, but the dummy *it* is not needed. The sentence would be improved by saying:

  > The primary aim of manifold theory is to classify topological manifolds.

- The verb structure in 3 is clumsy and prompts the reader to ask: *good* for whom? Typically the silent answer is *for the author* or *for maths* or *for the community*, but this is an opinion word that opens up questions.

- The verb in 7 is active, and could not have been made passive. The verb in 8 is passive. Active and passive can mix within the same sentence, but here a single noun *manifolds* is referred to both in with active voice as something *we are working with* and in the passive as something that *is assumed to be etc.* It does not sound right.

With that in mind, here is the original paragraph from the textbook *4-Manifolds and Kirby Calculus* by Gompf and Stipsicz' (p. 6), referred to from now on as [GS99]. There is a one-to-one correspondence between verbs her and those in the modified example above, but we have once again added the superscript numbers for ease of reference.

> Our primary aim in manifold theory is[1] to classify topological manifolds, i.e., to give a complete list of n-dimensional (closed) topological manifolds, and to find a way to tell which topological manifolds carry smooth structures (have C^∞-atlases). Furthermore, if there is[2] one such atlas, we would like[3] to determine the total number of these up to diffeomorphism. In most dimensions this aim cannot be achieved[4] for algebraic reasons (cf. Theorem 1.2.33 and Exercise 5.1.10(c)); in those cases we will impose[5] further conditions (like simple connectivity) for the manifolds at hand. For a better understanding of results concerning 4-manifolds, we will conclude[6] this section with theorems concerning manifolds of dimension different from 4. Assume[7] that the manifolds we are working[8] with are closed, connected and oriented.

As this is an excerpt from the textbook's introductory section, the authors have chosen to take a more friendly approach and use *we* frequently in order to engage the student. Hence, they talk about *our primary aim*, and they use active verbs to convey the goals of their book and their subject. Here are some specific observations:

- In 2: Their phrase *there is* is much stronger than the passive *is found*; the former being a statement of fact, the latter being a statement about a search conducted by an unknown entity.

- In 3: Their *we would like* is a much less contentious word choice than the passive *it would be good*, because even if the reader does not understand why something is *good* in general, they will have no trouble accepting the authors *would like* to determine something.

- In 4 and 5: This is an example of a sentence that has both a passive and an active voices that are compatible. It also hints at their appropriate applications: *cannot be achieved* talks about a general truth that is backed up by further references; *we will impose* talks about specific measures the authors will take to get around the difficulties. (We will address that distinction in the remainder of this section; read on.)

Finally, let us briefly discuss the verb used in 5, and how you would systematically go about analysing any problems and rephrasing it.

Whenever trying to work out the grammatical structure, strip the sentence to the a bare minimum. Here, take the second half of the sentence, after the semicolon:

> in those cases we will impose[5] further conditions (like simple connectivity) for the manifolds at hand.

and reduce it to its subject-verb-object frame:

> We will impose conditions.

Working with that, we can consider other options:

1. We *will impose* conditions. (*Active original.*)
2. Conditions *will be imposed*. (*Passive.*)
3. *Imposing* conditions *will be necessary*. (*Active nominalisation.*)
4. We *will be imposing* conditions. (*Active continuous.*)

The first two options we discussed above. Option 3 switches the passive to the active voice of the verb *to be* at the cost of converting the more "interesting" verb *impose* to a noun *imposing* (as well as requiring the word *necessary* to complete the grammatical structure). Whenever you can, avoid nominalisiation;[3] it deadens the writing. Option 4 is in the future continuous and should not be confused with the nominalisation (and should not appear in your writing, as discussed above).

Example: Writing maths details. The details are predominantly written in the active voice and in the indicative or imperative mood.

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[3]This is an example of sentence where a comma saves you two words. Observe:

- Whenever you can avoid nominalisation, do so.
- Whenever you can, avoid nominalisation.

Both sentence say the same thing. See Lecture 5 for more on commas.

Proof. Use the theorem to write M as ∂X, and decompose X as a handlebody. Then the union of 0- and 1-handles is $\sharp n S^1 \times D^3$. By surgery on circles in X, we can replace this by $\sharp n S^1 \times D^2$, so without loss of generality we can assume (after changing X) that X has no 1-handles. Similarly, we can eliminate 3-handles by turning the handlebody upside down and surgering $I \times M \sharp m S^1 \times D^3$. Now M bounds a 2-handlebody and hence the corollary follows.

[GS99, p. 159]

Note how *we* as subject does appear, but not with every verb and mostly in places where rephrasing would mean switching to the passive voice.

Example: Discussing generalities. The accent on generalised ideas is achieved by placing them first in the passive voice.

- Much time was spent on ambitious goals that gauge theory now shows are impossible. [GS99, p. xii]
- Another sort of structure frequently used by topologists is a *piecewise linear (PL-) structure*, which is defined by an atlas whose transition functions respect a suitable triangulation of $\mathbb{R}^n$. [GS99, p. 7]
- In general, the term *exotic smooth structure* is used to refer to smooth structures not diffeomorphic to the given one on a smooth manifold X. [GS99, p. 7]

Example: Discussing other people's work. It is unusual to use *they* (other authors) or *he/she* in mathematical writing unless it cannot be avoided; these pronouns draw too much attention to themselves. Thus it may be tempting to use only the passive, but the following example shows this does not have to be the case.

Remark 1.2.19. The proof of Proposition 1.2.18 given in [**W1**] goes as follows. The case $n = m = 2$ is proved first, by explicit construction of automorphisms. In this case it is also shown that if x is characteristic, it can be mapped to a canonical element depending only on $Q(x, x)$. <u>*This idea*</u> extends to general n and m, and proves Proposition 1.2.18 in the characteristic case. For x, y not characteristic, <u>*the proof of the $n = m = 2$ case*</u> provides an automorphism mapping the noncharacteristic vectors either into the subspace $2\langle 1\rangle \oplus \langle -1\rangle$ or into $\langle 1\rangle \oplus 2\langle -1\rangle$. Now for general n and m, the splitting $n\langle 1\rangle \oplus m\langle -1\rangle = (2\langle 1\rangle \oplus 2\langle -1\rangle) \oplus ((n-2)\langle 1\rangle \oplus (m-2)\langle -1\rangle)$ combined with induction gives the result in the noncharactersitc case. For details see [**W1**].

[GS99, p. 12]

The anatomy of **Remark 1.2.19**: first, the authors establish they are discussing someone else's work with a reference and with the passive; second, they switch to the active voice and refer to what was in already brought up by using the underlined words; third, they discuss the general case without any indication of authorship, but by now we understand they are following [**W1**]; and finally they give the reference again.

6.1.4 Omitting *that* and *that is/was*

Consider the following statement:

> We now understand the method used in the algorithm failed.

How did you interpret this sentence: did the method fail or the algorithm? Here is the statement again with its two interpretations. Read the terms in square brackets as part of the sentence.

A. We now understand the method used in the algorithm failed.

B. We now understand the method used in the algorithm [that] failed.

C. We now understand the method [that was] used in the algorithm failed.

In B, the reader plugs in the relative pronoun *that*, and this determines the sentence meaning: the algorithm failed. In C, the reader plugs in the *that was*, which makes for the passive voice *was used*, and this determines the sentences meaning the other way: the method failed. This is an example of "garden path effect"; the sentence has two valid interpretations that depend on which omissions are assumed by the reader. Both types of omissions—omitting *that* and omitting part of the passive—are valid and done all the time.

The three examples we used above to illustrate the application of the passive voice also showcase these forms of omission (and inclusion):

1. Much time was spent on ambitious goals [that] gauge theory now shows are impossible. [GS99, p. xii]

2. Another sort of structure [that is] frequently used by topologists is a *piecewise linear (PL-) structure*, which is defined by an atlas whose transition functions respect a suitable triangulation of $\mathbb{R}^n$. [GS99, p. 7]

3. In general, the term *exotic smooth structure* is used to refer to smooth structures [that are] not diffeomorphic to the given one on a smooth manifold X. [GS99, p. 7]

In example 1, the authors actually did include *that*, though it could have been omitted. One reason for their choice might have been that both *goals* and *gauge theory* start with a *g*; another reason might have been the useful pause provided by the word *that*, which allows the reader to interpret the sentences more easily.

In example 2, part of the passive form (*is used*) is omitted. (This example is also a rare case where humans are the object of a passive sentence.)

In example 3, we are faced with a third kind of omission: *that are* is not part of a passive construction. Its omission is common in English. Consider the pairs:

- I attended a lecture *that was* on Hilbert manifolds.
- I attended a lecture on Hilbert manifolds.
- I'm attending a lecture *that is* on Hilbert manifolds.
- I'm attending a lecture on Hilbert manifolds.
- Yesterday's lecture, *which was* on Hilbert manifolds, taught me a lot.
- Yesterday's lecture on Hilbert manifolds taught me a lot.
- Today's lecture, *which is* on Hilbert manifolds, should be interesting.
- Today's lecture on Hilbert manifolds should be interesting.

Note how the first part of the sentence determines the tense, and therefore the second verb (*is/was*) does not have to and can therefore be omitted. Furthermore, note that when removing the verb from the non-restrictive relative clause (starting with *which*), you also have to remove the commas.

6.2 Abusing Grammar

Your intentions as a writer may be sound, but sometimes the sentences get out of hand accidentally: you change your mind halfway through; you think to add a few more adjectives; you chop up the information to help the reader or lengthen it to lump all the relevant information together. It's all in a days work, but here are a few things to look out for.

6.2.1 No: mismatching numbers

Sometimes a sentence might begin with a view towards talking about a single subject, but then the author decides to add another subject and forgets to change the initial noun to plural.

- BAD: The *group* G_1 and G_2 have no torsion subgroups.
- OK: The *group* G_1 and the *group* G_2 have no torsion subgroups.
- GOOD: The *groups* G_1 and G_2 have no torsion subgroups.

Sometimes the other words subvert the subject of a sentence, and the verb does not match the actual subject in number.

- BAD: Our chief aim in defining the function f on the cartesian product of the topological spaces $\{X_i\}_{i\in I}$ *are* to study the behaviour of ...
- GOOD: Our chief aim in defining the function f on the cartesian product of the topological spaces $\{X_i\}_{i\in I}$ *is* to study the behaviour of...

Here, even though the last noun before the verb is plural (*spaces*), the subject is the singular (*Our chief aim*).

6.2.2 No: Dangling the participle

Participle phrases start with the present or the past participle of a verb (e.g. *proving* or *proved*), and they describe a noun or pronoun. For example:

- *Assuming the conjecture is true*, we can show...
- *Developed fully*, this theory might be able to provide some solutions.

In maths writing, the former is seen more frequently than the latter, but neither type occurs often.

When writing participle phrase, take care that there is confusion about the noun or pronoun to which the participle pertains. For example, this is incorrect:

> *Differentiating* $(x-1)^3$ *twice*, the saddle point is shown to be at $(1, 0)$.

Differentiating is not an action done by the saddle point. To fix the grammar issue, you could write any of the following:

1. Differentiating $(x-1)^3$ twice, *we* find that the saddle point is at $(1, 0)$.
2. *By* differentiating $(x-1)^3$ twice, we find that the saddle point is at $(1, 0)$.
3. Differentiating $(x-1)^3$ twice *reveals* the saddle point is at $(1, 0)$.

Option 1 uses the present participle phrase correctly. In option 2, the preposition indicates the method. Option 3 is an example of nominalisation that then requires a fancy verb—such phrasing may be acceptable in the introduction to some high-level description of a complex technique (so not differentiation!), but in all other cases, options 1 and 2 are preferred.

6.2.3 No: overburdening the pronoun

This is a grammatically correct sentence, but it pins a a lot of diverse data on a single pronoun.

> Let M be a 3-dimensional manifold. Suppose that *it* has a non-trivial fundamental group, a torsion-free second homology group, a taut foliation $\mathfrak{F}$, and a Heegaard splitting as follows.

Break up such data-dumps into separate sentences.

6.2.4 No: Overburdening the noun

This sentence is difficult to parse because a reader needs to determine which of the possibly unfamiliar words is the noun.

> A simply connected, minimal, geometrically ruled surface is bihomorphic to a Hirzebruch surface $\mathbb{F}_n$.

Such a condensed form might be useful occasionally (and for experts) but if you are going to talk about these surfaces, you might as well introduce the notation in time, and ease the reader into the matter.

> If S is a simply connected, minimal, geometrically ruled *surface*, then S is biholomorphic to a Hirzebruch surface $\mathbb{F}_n$. [GS99, p. 88]

Even though the number and order of adjectives is the same, the (English-speaking) reader will parse a sentence more easily if first given the symbol S and the verb *is*.

6.2.5 No: Sentence trains or grains.

With technically difficult assumptions, do not feel compelled to put them in one sentence train—even if the punctuation and grammar are correct. The star marks the start of a new sentence; the underlined words connect clauses where new sentences could have started.

> Let Y_1^n, Y_2^m be transversally intersecting, connected, smooth submanifolds of complementary dimensions in the simply connected $(n+m)$-manifold X^{n+m}, *and* assume furthermore that $m \geq 3$ and $n \geq 2$, *unless* $n = 2$, *then also assume* $\pi_1(X - Y_2) = 1)$, *and let* $p, q \in Y_1 \cap Y_2$ be intersection points with opposite signs. $\boxed{*}$ Then there exists an isotopy $\varphi_t(t \in [0,1])$ of $\mathtt{id}_X$ such that $\varphi_(Y_1) \cap Y_2 = Y_1 \cap Y_2 - \{p, q\}$.

Likewise, if you have pitched your material to the correct audience[4], then you should not have to spoon-feed them information like grains of sugar.

> Let Y_1^n, Y_2^m be smooth submanifolds of complementary dimensions in the simply connected $(n+m)$-manifold X^{n+m}. $\boxed{*}$ *Let them also be* transversally intersecting and connected. $\boxed{*}$ Assume furthermore that $m \geq 3$ and $n \geq 2$. $\boxed{*}$ *In the case when* $n = 2$, suppose $\pi_1(X - Y_2) = 1$. $\boxed{*}$ If $p, q \in Y_1 \cap Y_2$ are intersection points with opposite signs, then there exists an isotopy $\varphi_t(t \in [0,1])$ of $\mathtt{id}_X$. $\boxed{*}$ *This isotopy has the following property* $\varphi_(Y_1) \cap Y_2 = Y_1 \cap Y_2 - \{p, q\}$.

What follows is the original. Note how it makes use of one sentence to set up the main assumption, a second sentence to give an auxiliary assumption (and a parenthesis to offer an auxiliary assumption of an auxiliary assumption!), and a third *if..., then...* sentence to give the content of the theorem.

> **Theorem 9.2.7.** (Whitney trick) Let Y_1^n, Y_2^m be transversally intersecting, connected, smooth submanifolds of complementary dimensions in the simply connected $(n+m)$-manifold X^{n+m}. $\boxed{*}$ Assume furthermore that $m \geq 3$ and $n \geq 2$ (and when $n = 2$, $\pi_1(X - Y_2) = 1$). $\boxed{*}$ If $p, q \in Y_1 \cap Y_2$ are intersection points with opposite signs, then there exists an isotopy $\varphi_t(t \in [0,1])$ of $\mathtt{id}_X$ such that $\varphi_(Y_1) \cap Y_2 = Y_1 \cap Y_2 - \{p, q\}$. [GS99, p. 348]

> A grammatically correct sentence is not necessarily a sentence friendly to a mathematician's ear. Be friendly (and sensible).

[4]Understanding your audience will be discussed in the following lecture.

Problem Sheet 6

PROBLEM 1. Choose the sentence with the best verb form.

A. We will say that an even-dimensional manifold is *symplectic* if...

B. We say an even-dimensional manifold is *symplectic* if...

C. It has been said that an even-dimensional manifold is *symplectic* if...

D. We have said that an even-dimensional manifold is *symplectic* if...

PROBLEM 2. Choose the sentence with the best verb form.

A. In this paper we will be showing that...

B. In this paper we shall be showing that...

C. This paper will be showing that...

D. This paper will go on to show that...

E. This paper will show that...

F. This paper is to show that...

PROBLEM 3. Can the material in square brackets be removed to leave a grammatically correct sentence (even though in writing you may choose not to remove it)? Give a *yes* or *no* answer in each.

A. My thesis describes a problem [that was] conjectured over a century ago.

B. [That] is a problem which cannot be solved.

C. That is a problem [which] cannot be solved.

D. The problem, [which was] once thought to be difficult, is nowadays given to undergraduates as an exercise.

E. It has been shown [that] this condition is redundant.

PROBLEM 4. Choose the options which have "dangling" phrases and are therefore grammatically incorrect.

A. Acting on the set S, we find that the group G...

B. Substituting (1) into (3), the differential equation becomes...

C. Removing the redundancies, we find that...

D. When constructing the example, the proof illustrated that...

E. Proofreading step by step, a gap in the proof was found.

F. Proofreading step by step, a gap in the proof was found by us.

G. Proofreading step by step, we found a gap in the proof.

LECTURE 7

Reader and Writer

'I'd do anything for you.'
'Would you please please please please please please please stop talking?'

— Ernest Hemingway, *Hills Like White Elephants*

Even technical writers write to be read and not so someone can tell them to shut up.

In this lecture we discuss who your "Ideal Reader" is and how to pitch your work so that they best appreciate it. More specifically, we discuss how to determine your Ideal Reader depending on the type of text your are writing (Section 7.1), and how to fine-tune your work accordingly (Section 7.2). We then turn to the writer's presence—which should be felt, but not overwhelmingly so—and how the writer is meant to address themselves in a text (7.3). Finally, we give some self-editing tips that should improve the readability of your text (Section 7.4).

7.1 The audience and the Ideal Reader

Mathematical truths exist without us, but if we already chose to study them and convey them to other humans, we ought to conform to the basic tenets of clear, precise, and *directed* communication. That is, we have to always consider the existence of a reader—a person who will be spending their precious time trying to understand our words, and we have to tailor our work to suite them.

The following is a useful analogy to have in mind:

1. A person stands on a platform saying the same thing to every passerby. The passersby understand nothing, care not, and walk on.
   This is the mathematician who writes for nobody in particular, and therefore nobody in particular reads their work.

2. A person stands on a platform talking to themselves. Most passersby walk on, but even if someone stops to listen, they understand nothing because the speech is only intelligible to the speaker.
   This is the mathematician who writes for themselves and therefore nobody else can understand their work.

3. A person invites all their friends from work, stands on a platform, and addresses them: "Listen everyone, I'l tell you something that'll tickle your fancy."
   This is the mathematician who writes for a particular set of people, aiming to have all of them pay attention.

You always want to be that third type of mathematician, directing your communication at an ideal audience.

7.1.1 Who is your audience?

Solid technical foundations and language prowess are two aspects of a maths text. The former is up to you; and so far these lecture notes have focused on offering guidelines to the latter. The third, understated aspect that crucially slants each piece of writing is the *pitch*.

Most of the time you need not behave like a journalist or fiction writer who has to pitch their work to a grumpy audience, but you should be writing for the same reasons: to be read, to be understood, and—let's face it—to be appreciated. And appreciation comes in many forms: as a grade, as an acceptance letter, as a coveted research position or industry job, or simply as the pleasure of having your ideas contribute to humanity's knowledge of the sciences.

This is where directing your work at a concrete audience, or *pitching*, comes in.

An appropriate pitch excludes what the audience already know, and presents the rest of the material in a relevant way. Since you cannot pitch to a motley bunch, your audience needs to be homogeneous, but once your audience is homogeneous you can pick one member, any member, call them the Ideal Reader and think of them when writing. Will they understand what you are saying? Will they be familiar with that reference or that argument? Or will they find what you are writing obvious and tedious?

To answer those questions, you need to determine two things:

(i) What is the background of your Ideal Reader?

(ii) What is their mathematical maturity?

Here are some extreme examples:

- A fifty-year-old professor of geometry probably has "infinite" mathematical maturity, but no background on a specialised field of statistic.
  This means you need to give specialised definitions but do not need to explain any mental "tricks" or standard mathematical patterns of thought.

- If the same professor happens to be the founder of your specialised field, then they[1] have both infinite mathematical maturity and infinite background.
  This means you need to explain only the very latest, cutting-edge, original research that you have been doing.

- A whizz-kid of sixteen might have specialised knowledge of algorithms, but very little mathematical maturity.
  This means that you need to focus on explaining the "big picture" and on mathematical patterns of thought that are gained only through experience.

[1]You may have noticed that we write *they* as a pronoun for a singular subject. The alternative would be to use *he*, *she*, or *he/she*, which are all gendered pronouns. Using *they, their, themselves* is a gender-neutral way of referring to a single person.

- First year undergraduates usually have neither the background nor the maturity in any field.
  This means you need to start from scratch: specialised definitions and patterns of thought.

As you are starting to realise, the Ideal Reader resides in your head (as do all ideal things in maths). Let us look at some examples of audience and pitch in practice.

7.1.2 Theses and semester projects

At the ETH there are four rungs of student work: the semester project, the bachelor's thesis, the master's thesis, and the PhD thesis, ordered from least to most demanding. Most of these will be read only by an adviser who is already familiar with the material. Therefore, the adviser is not your Ideal Reader.

Your Ideal Reader is someone of similar mathematical background to yourself as you were *before* you started the project or thesis. When writing, think of a colleague who is an acquaintance, but not a close friend, and think of how you would quickly and clearly explain your work to them. Inject some animation and enthusiasm in your work, especially in the beginning to get them intrigued, but do not overdo it, for they too are professionals: they care primarily about the maths.

The role of the adviser as the actual reader of your work is to decide whether you have done a reasonably good job at addressing your Ideal Reader. You are graded not for the maths you learned, but for the maths you were able to pitch correctly and explain coherently.

For example, this means that if you wish to discuss symplectic geometry in a third-year semester project, you need to define a symplectic manifold, but you can assume the Ideal Reader is familiar with sets, topological spaces, calculus and so on. In contrast, in a PhD thesis on the same subject you would assume the Ideal Reader is familiar with the basics of symplectic manifolds and you would spend only a paragraph stating a few of the standard results in the area.

> Your Ideal Reader is a student in your field.

7.1.3 Preprints

Starting at the level of a Master's thesis, and certainly that of a PhD, you work may be read by a wider audience and not just your adviser. This changes little in your approach: you are still writing for a colleague who is at the same level as you were prior to commencing the work.

As you mature mathematically and your work becomes more specialised, your Ideal Reader will mature with you and also specialise up to a point. You will also start taking other considerations into account when imagining your Ideal Reader. If your preprint is primarily in number theory but you think it might have some applications in statistics, you might pitch your writing so it includes sufficient number theoretical background that statisticians can read and understand the preprint too. Your goal is to increase your core audience while staying relevant. Remember scenario 1 from above: you do not want to be speaking to the whole world, nor speaking only to yourself. This is a classical maximisation problem.

> Your Ideal Reader is a colleague in your field.

7.1.4 Application essays

Applications for further studies may require a motivational essay or pesonal statement, explaining your education record and your hopes for the future, as well as, giving an overview of your maths-related skill in somewhat non-technical terms. These essays will be read by members of the application committee, who are professional scientists, though perhaps not in your precise field or interest area. Before writing the essay, your task is to determine, as closely as possible, who will be reading it.

For example, in mainland Europe you generally apply to a particular professor with which you wish to do your PhD; in North America and the United Kingdom, you apply to a university department. In the former case, anything you write should be directed the person you are applying to (your Ideal Reader is your desired future adviser who will be impressed by a specific, advanced theorem you name-dropped); in the latter case, the motivational essay should be kept general (your Ideal Reader is a general mathematically minded scientist who will not be impressed by a specific, advanced theorem you name-dropped).

> Your Ideal Reader is the person who will be reading the essay.

7.1.5 Talks

Talks are not read by an audience, but delivered to it. That said, a formal seminar or colloquium talk still has to be prepared, usually written out by hand or digitally, and it most definitely has to be pitched at the correct level. Suppose you do not pitch it correctly. Suppose you enter a room of ten-year-olds and start telling them about Banach manifolds. You will get puzzled looks, questions, and eventually laughter. Any prepared slides or materials will prove useless. You will have to invent, on the spot, a new talk and it will not be fun or productive.

So when preparing a talk you should figure out who your Ideal Listener will be. It is a graduate seminar? Is it a group seminar? A department seminar? A job interview? A plenary session at the ICM?[2]

> Your Ideal Listener is a typical attendee of the seminar series you are invited to speak at.

[2]You wish! If you are reading this before giving a plenary talk at the ICM, please stop immediately.

7.2 Addressing the Ideal Reader

Once you have determined your audience, imagined your Ideal Reader, and decided how to pitch your work, there still remain two closely connected questions:

(i) How do your address your Ideal Reader?

(ii) And how do you refer to yourself in writing?

We discuss the answer to (i) here, and the answer to (ii) in Section 7.3.

7.2.1 Setting the tone

In a formal piece of writing you generally do not address the reader directly, though your aim is to establish a professional tone. This means you want to be courteous, but not machine-like rigid, and you want to be friendly, but not gushing with camaraderie. Therefore, you avoid the following extreme cases:

- Let f be a smooth function. Let X_f be the set stationary points of f. Let g be a smooth function. Let X_g be the set stationary points of g. Suppose $X_f \subset X_g$. Let h be a smooth function. Let...
  (*The machine.*)

- Let's think about these smooth functions $fandg$ and their stationary points. You must remember how cool a concept this was back in high-school, when we would draw a graph and look for its "special points". But now we have better tools to study graphs. Hey we do not even have to draw a graph! We can just differentiate.
  (*The enthusiastic best friend.*)

Establishing a professional tone also means avoiding statements that are loaded with more subtle emotion or attitude. We have already mentioned some of these examples in passing:

- The remainder of this proof is obvious if you just think about it.
  (*Condescending.*)

- We believe the reader is smart enough to complete the proof.
  (*Patronising.*)

- My proof is as sleek as Riemann's.
  (*Snobbish.*)

- After much lucubration I have alighted on this most elegant of proofs.
  (*Pretentious.*)

- Where every other version of this theorem requires an additional assumption we require none, leading to a shorter, slicker proof that can be explained to anyone with a basic grasp of graph theory.
  (*Bragging.*)

- Most readers should by now have worked out the end of the proof.
  (*Disdainful.*)

And so on and so on. The sins of a tone-deaf writer can take many forms.

> Aim for an amicable, but strictly professional tone.

7.2.2 Where tone meets pitch

All the examples above have at their core an emotion, attitude, or opinion that does nothing to further the reader's understanding of the content. Conversely, if such content *does* further the reader's understanding it should be included. For example, you might say:

> It is *surprising* that in four dimensions there are infinitely many exotic spheres.

This is not an emotionally loaded statement so long as you go on explain why it is surprising, and so long as this information could be surprising to your Ideal Reader. The inclusion of such a word then draws their attention to a mathematically salient point. If you are writing for fifteen-year-olds, you might include more such words. If you are writing for researchers in the area, they would not understand why you are finding this fact surprising.[3]

On the other hand, some ordinary English words that should be considered informal have taken on a special, coded meaning in mathematical English (e.g. *favourable*, *appropriate*, *tweaking*, etc). In textbooks or lectures notes you often come across statements that talk about the bigger picture, but have not yet developed enough machinery to be precise, or the needed machinery is beyond their scope. For example:

> Using Kirby calculus one can (under *favourable* circumstances) prove that 4-manifolds defined by different constructions are actually diffeomorphic. [GS99, p. 18]

Here the word *favourable* is used as a shortcut to mean *we would need to append certain conditions that are cumbersome or you have not yet studied*, but note how it *does* further our understanding of the maths: we immediately know the statement is not as straightforward as it looks and that it is actually false in many cases. Understanding the meaning of these "hedge words" and being able to use them correctly is one hallmark of mathematical maturity.[4]

In high-level maths, hedge words are indispensable shortcuts, and theses, papers, and application essays all use them to a certain degree. However, we caution that hedge words should only be used when precision is beyond the scope of the text or when the pitch of the text requires staying away from technical details.

[3] *Surprise* by definition has to be unexpected.

[4] When we gave an example above of an Ideal Reader with "infinite" mathematical maturity, what we meant was that such a reader, given the specialised definitions of an unfamiliar field, would be able to infer from a text replete with hedge words the essential achievements and problems of that field.

Do not convey attitude, opinion, or emotion, unless doing so would further the reader's understanding of the mathematical content. Use hedge words only when restricted by scope or pitch.

7.2.3 Telling the Ideal Reader what to do

Now let us zoom in on the actual mode of address. How do you tell the reader to pay attention?

Firstly, in Lecture 6 we discussed that maths often uses the imperative mood of a verb:

- *Let* k tend to infinity.
- *Set* $n := 2$.
- *Suppose* that the conjecture is true.

The imperative mood may seem like it is commanding the reader to do these actions, when of course it is not; writing in this way is merely a convention. However, in a sense, the imperative does create a relationship between the writer who is exhibiting the piece of mathematics and the reader who is following along, more so than the indicative (which simply states facts).

Secondly, depending on the style of a text, you may have come across a liberal application of the personal pronoun *you*. This pronoun is a natural extension of the imperative mood:

- If *you let* the sum tend to infinity...
- When *you set* $n := 2$...
- Therefore, *you could suppose* that...

Again, this is merely part of the conventional vocabulary used primarily in less formal writing, in learning materials, and in speech. In modern texts *you* is understood to be a warmer, more humane version of the pronoun *one* (*one lets, one sets, one supposes*), but is not recommended in formal writing.

When writing formal, technical mathematics avoid appending *you* to verbs in the imperative mood.

7.3 The writer's persona

Striking the correct tone when addressing the Ideal Reader involves choosing the correct presence for yourself within the text. Recall from Lecture 6 that you have these four options when writing maths sentences:

1. The active voice, with yourself as the subject.

2. The active voice, with a maths term as the subject.
3. The passive voice, with yourself as the object.
4. The passive voice, with a maths term as the object.

As mentioned above, you should balance between sounding machine-line, which would mean using mostly Options 2 and 4, and sounding overly friendly (or self-centred), which would mean using mostly Options 1 and 3.[5] However, even if you decide on that balance, the question remains in Option 1 how to you refer to *yourself*. Here are your choices:

(i) *I* prove the theorem in the Appendix.

(ii) *We* prove the theorem in the Appendix.

(iii) *The author(s)* prove the theorem in the Appendix.

(iv) *One* proves the theorem in the Appendix.

Let us discuss each choice individually.

(i) Using the first person singular may seem the most natural if you are the single author of a text. However, in mathematics using *I* is distracting: it draws the readers attention away from the content. It is best to reserve using *I* for a motivational essay where you discuss your personal history or for a dedication at the beginning of a formal work.

(ii) Using the first person plural may seem an odd choice for a single author of a text, but regardless of the number of authors, it has become the norm. Traditionally, *we* is taken to mean *writer and reader* and is therefore a less off-putting position to be in as a struggling reader (the burden of the struggle is shared, so to speak). However, nowadays this is simply the option everyone expects, and so it is least obtrusive in a text.

(iii) Referring to yourself in third person is useful in situations where you must differentiate yourself from the reader and would be much tempted to use *I*; this is usually in situations where you must humble yourself and admit your ignorance in some respect. For example, in a conclusion, you might say:

> At this time *we* do not know how to prove the Conjecture, but *we* are hoping to tackle it using a modification of the methods developed in Section 4.

The reader will not misunderstand what is being said, but it would also be acceptable to emphasise the researcher's role at this stage and say:

> At this time *the author* does not know how to prove the Conjecture, but *is* hoping to tackle it using a modification of the methods developed in Section 4.

[5]Though, option 3 is difficult to pull off and not sound ridiculous; we saw one good application in the previous lecture and it was of the form: *Such methods are often developed by topologists....*

In this version the other *we* was omitted because the third person is only really appropriate for sentences that can avoid having to repeat the subject; you do not want to have to write a paragraph saying *the author this, the author that*, and the last thing you want is to start using pronouns like *he* or *she*.

(iv) Sentences using the indefinite pronoun *one* are impersonal and formal. They feel halfway between the active voice (which they utilise) and the passive voice. In everyday English—unless you are careful—the use of *one* can sound a bit pretentious:

> <u>*One*</u> might develop this argument further, but <u>*one*</u> could not be bothered.

Or it can sound quaint.

> <u>*One*</u> is careful when developing such an argument, because so many other papers have failed at the task.

However, occasionally *one* can come in handy in a formal setting where you might be tempted to address the Ideal Reader using the less formal *you*, but it is not appropriate to do so.

> <u>*One*</u> can think of the 3-dimensional sphere S as union of two solid tori T_1 and T_2.

When writing formal, technical mathematics refer to yourself as *we*.

7.4 Being kind to your audience

Whenever possible you should ease a reader's task. Write shorter paragraphs, rather than longer. Use a parallel construction to emphasise the parallel nature of the content. Remove unnecessary *to be* statements and use stronger, more precise words. Proofread and check for bad breaks. It takes a little bit of work, but makes a vast difference to the presentability of your text. In other words, make your text readable so your readers can appreciate its ideas.

7.4.1 Paragraphing

Aside from specific, labelled chunks of mathematical writing, such as *theorems*, *lemmas*, *proofs* and so on, the standard unit of thought is the paragraph. But what is a unit of thought? Without philosophising, the end of a paragraph is best considered as a pause longer than a period—a pause you would make to give a friend time to process what you have said.

As in speech, torrents of words are difficult to process on the page. A solid block of text looks tedious and daunting, and a paragraph break symbolises a mini-break for the reader. So paragraph as often as is possible while retaining clusters of sentences that talk about the same subject.

7.4.2 Parallel structures

Parallel structures are much easier to follow.

- On paragraph level, exposition can be structured via the phrase-patterns *first, next, finally* or *first, second, third* and the like.
- On sentence level, a colon and semi-colons can be used to list items in a way which makes their ordering and similarity apparent.
- In between the two levels, there is a sentence-to-sentence parallelism. Sentences can be made parallel grammatically so that their content can be contrasted.

    > *In theory*, the statement *is* unambiguous about its meaning. *In practice*, its meaning *is* garbage.

    For another example see the opening paragraph of Section 7.4, where we use a parallel structure in four sentences starting with *Write, Use, Remove, Proofread* to highlight their similarity.

7.4.3 To be or not to be

The verb *be* can be powerful and can be a powerful bore. Its power comes from it being a *copula*—a word connecting a subject and complement. (There are about thirty others, but none are as fundamental, e.g. *seem, appear, smell, taste, go, feel.*) A demonstration of power:

> I am a mathematician.

On the other hand, *be* is versatile and can be used in most sentences. A demonstration of boredom:

> There *is* a continuous function g such that there *is* one root of g and such that the derivative of g *is* either +1 or -1 away from that root. It *is* clear that one such example *is* the absolute value function $g(x) := |x|$.

We have mentioned before that words such as *clearly, actually,* and other adverbs should be avoided unless justified. Our advice was to search for words ending in *-ly* to check whether any of them had sneaked in unintentionally. You can also try searching for patterns such as *there is/there are*, *this is*, *it is*, and so on. Sometimes you will find redundancy:

- There are several proofs that are valid.
  (*Redundant verb.*)
- There are several valid proofs.
  (*Neater.*)

Sometimes you will find that you can improve the precision:

- That is how the preceding lemma will help us understand...
  (*The antecedent of* <u>*that*</u> *may not be clear.*)

- Analysing the handlebody decomposition described in the preceding lemma will help us understand...
  (*Specifies what will be helpful.*)

Sometimes you will find the passive that can be made active if more information is provided (though you may not wish to always make it active).

- It is proved by induction....
  (*Not necessarily bad, but it should not be a sign of imprecision or laziness because you could not remember a reference.*)

- The authors in [YJ09] prove the result by induction.
  (*Precise and helpful.*)

There are is versatile: there are cows that can jump over the moon (in our imagination) and there are mathematicians around the world (and possibly in another galaxy). So whenever *there are* is understood to mean *there exists*, which is closer to the mathematical symbol $\exists$, it is preferable to use the latter. Especially when the object in question is specifically a mathematical construct. For example:

- There are several valid proofs.
  (*A general statement that will likely be further elaborated.*)

- There exists a function g such that...
  (*A precise statement where it is better to use* <u>*exists*</u> *than* <u>*are*</u> *because it is a mathematical object.*

7.4.4 Ordering words for flow

We have talked about the different choices you have when deciding how to phrase a sentence. But once you have made those choices (active, passive, subject, object, tense etc), you are left to decide how best to order the words to make the sentence as readable as possible. Unusual punctuation (too much or too little) breaks up the flow of the sentence. The same goes for certain phrases: you could put them in a number of places, but some places are better than others. This becomes apparent if you try to reorder a sentence, or even just displace a couple of words.

For example, consider the following sentence on the topic of symplectic diffeomorphisms taken from the *Princeton Companion to Mathematics* (p. 298, by Gabriel P. Paternain).

> Since symplectic linear transformations have determinant 1, we can conclude using several-variable calculus that a symplectic map is always locally volume preserving and locally invertible; roughly speaking, this means that the map $\phi\colon A \to \phi(A)$ is invertible whenever A is a sufficiently small subset of U, and $\phi(A)$ has the same volume as A.

Now isolate the second part of the sentence, and try shuffling around the phrase *roughly speaking*. Note the changes in sentence flow.

(i) *Roughly speaking*, this means that the map $\phi\colon A \to \phi(A)$ is invertible whenever A is a sufficiently small subset of U, and $\phi(A)$ has the same volume as A.[6]
(*This the original. The phrase is unobtrusive, it refers to the whole remainder of the sentence, and it warns the reader that what they are about to read is not a completely accurate statement.*)

(ii) This means, *roughly speaking*, that the map $\phi\colon A \to \phi(A)$ is invertible whenever A is a sufficiently small subset of U, and $\phi(A)$ has the same volume as A.
(*The sentence is now of the form: two words, then a comma, another two words, then a comma, anther three words, then a formula, etc. This version is choppier than the original.*)

(iii) This means that the map $\phi\colon A \to \phi(A)$ is invertible, *roughly speaking*, whenever A is a sufficiently small subset of U, and $\phi(A)$ has the same volume as A.
(*Here the reader starts of thinking that the definition will be precise, but then meets the phrase halfway through the sentence and needs to readjust expectations. Also the phrase severs the sentence in two.*)

(iv) This means that the map $\phi\colon A \to \phi(A)$ is invertible whenever A is a sufficiently small subset of U, *roughly speaking*, and $\phi(A)$ has the same volume as A.
(*The phrase now appears between two conditions of the statement creating confusion; the reader will wonder to what part of the sentence the phrase applies.*)

(v) This means that the map $\phi\colon A \to \phi(A)$ is invertible whenever A is a sufficiently small subset of U and $\phi(A)$ has the same volume as A, *roughly speaking*.
(*The reader is only told at the end of the sentence that the statement was imprecise; alternatively the phrase may seem to modify only the condition about volume.*)

7.4.5 Ordering words for emphasis

Whenever you can, you should reorder your sentence so that the words of greater importance come first (which is the location of greatest importance in a sentence). Consider these examples and how the emphasis changes with the word order.

A. An important example of a knot is the trefoil.
(*The focus is on a special kind of knot.*)

B. The trefoil is an important example of a knot.
(*The focus is on the trefoil.*)

[6] This is an example of a so-called *bad break*, where a syllable or symbol is carried over onto a line where it stands by itself. Below, in Section 7.4.8, we advise you remove bad breaks. However,here we are quoting a line verbatim and cannot therefore modify it to remove the bad break.

As the complexity of a sentence grows, more variations emerge. What you decide to emphasise is up to you but this decision will drive the reader's attention on a small scale (which will be felt cumulatively).

A. The trefoil is the only knot with crossing number 3.
   (*The focus is on the trefoil*)

B. Only the trefoil knot has crossing number 3.
   (*The focus is on the special properties of the trefoil, on its uniqueness.*)

C. The only knot with crossing number 3 is the trefoil.
   (*The focus is on knots with crossing number 3.*)

7.4.6 Selecting the concise expression

In Lecture **??** we saw that the words *that* and *that is/are* can be omitted from a sentence in some cases. Here are a few cases where a more concise expression is possible, and a few where it is not.

The coupled fragments are mostly interchangeable:

- the meaning [that] we intended
- our intended meaning
- the function's continuity
- the continuity of the function
- the dimension of the manifold
- the manifold's dimension
- the leaves of the foliation
- the foliation's leaves
  (*This would be understood, but is not common.*)

However, sometimes compression is not possible:

- GOOD: the category of small categories
- BAD: the small categories' category
- GOOD: the space of functions
- BAD: the functions' space
- GOOD: the set of points
- BAD: the points' set

If unsure, check the web for precedents.

7.4.7 Selecting the appropriate level of specificity

Whenever referring to an object or concept choose the most specific word that fits your context. Do not talk about an author's *work* if you mean a *theorem* from their latest paper; mention the theorem. Likewise, do not talk about an author's *theorem* if it you mean their whole *paper*, which includes many theorems that are relevant to your discussion.

- Their <u>*work*</u> led to the discovery of...
  (*This could mean anything taken up from the span of their careers. A useful statement for introductions and historical previews.*)

- The authors proved the result in their first joint paper <u>*paper*</u> [US42].
  (*This specifies the work, which might not say much if the paper is a hundred pages long and you are thinking of a lemma in the Appendix.*)

- Applying [US42, <u>*Theorem*</u> 4], we complete the proof....
  (*A specific, useful reference.*)

- Using induction on the number of summation elements, an <u>*argument*</u> first presented in [US42, Theorem 4]), we can then...
  (*An even more specific reference to the type of argument.*)

- The following <u>*inequality*</u> was derived in [US42, Theorem 4]:...
  (*Most specific kind of reference where you not only specify the location of the inequality, but also copy it out for the benefit of the reader.*)

Vagueness is troubling, but so is going to the other extreme and citing the lemmas of a paper or discussing the method of a proof when you should be taking a more high-level approach. So how do you decide on the level of specificity? Specificity it part of the pitch: you should be steering your decisions based on how much detail you think your Ideal Reader needs.

7.4.8 Removing bad breaks

After compiling a LaTeX file you may find certain visually unappealing details have appeared. Here is a sample:

- Bad break: a syllable is carried over into the next empty line.

  > The primary aim of manifold theory is to classify topological man-
  > ifolds.

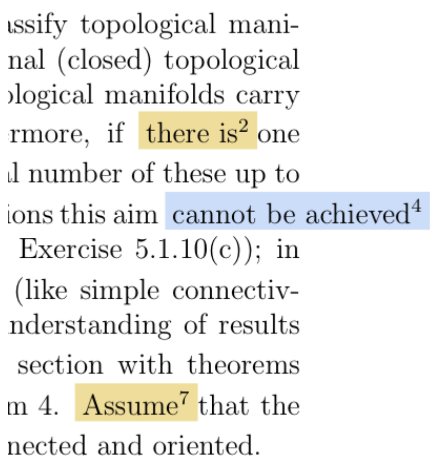
- Worse bad break: a single digit and square bracket carried over.

  > The term *exotic smooth structure* refers to smooth structures not
  > diffeomorphic to the given one on a smooth manifold X. [GS99, p.
  > 7]

- Worse bad break still: a rogue maths symbol and a reference appear together on a line.

> The term *exotic smooth structure* is used to refer to smooth structures not diffeomorphic to the given one on a smooth manifold X. [GS99, p. 7]

- The worst bad break: part of the text is jutting outside of the margin.


ıssify topological mani-
nal (closed) topological
ılogical manifolds carry
rmore, if there is[2] one
ıl number of these up to
ions this aim cannot be achieved[4]
Exercise 5.1.10(c)); in
(like simple connectiv-
nderstanding of results
section with theorems
m 4. Assume[7] that the
nected and oriented.

Most bad breaks can be removed by simply rephrasing sentences (though sometimes you have to tinker with the LaTeX, more on this in a later lecture).

7.4.9 Do not try to stand out

Mathematics relies on conventional phraseology and conventional appearance of material to ease the transmission of ideas. In previous lectures we have already warned against creative writing; here let us warn against odd or eye-catching formatting. It may seem like a redundant warning, but once you discover the power of LaTeX, you might be tempted to make your work stand out by using customised formatting. This is unprofessional and it draws the reader's attention away from the ideas in your work.

In semester projects, theses, applications, or preprints please refrain from:

- making the margins an unusual size,
- making the font an unusual size or an unusual typeface,
- using the **bold** typeface (unless it is for a title),
- underlining words,
- using CAPSLOCK,
- using *italics* for anything other than definitions of mathematical objects (and almost never for emphasis),
- using nonstandard symbols ❄ ➾ ♥ or fancy vignettes to separate sections,

- doing anything regarding style that has not already been done by your elders and betters. If you cannot point to a single illustrious precedent, do not do it. If you can point to one such quirky precedent, before emulating it, ask yourself whether you have sufficient mathematical cachet to pull it off. (The answer is invariably no.)

How to be kind to your audience.

1. Paragraph often.
2. Use parallel structures where appropriate.
3. Spellcheck.
4. Proofread.
5. Make sure your references point to the correct places.
6. Remove bad breaks and any other visual weirdness.

Problem Sheet 7

PROBLEM 1. Choose the most elegant statement that emphasises the importance of the *trefoil* (rather than other parts of the sentence).

A. The trefoil is the simplest nontrivial knot.

B. The simplest nontrivial knot is the trefoil.

C. The nontrivial knot that is the simplest is the trefoil.

D. The simplest knot that is nontrivial is the trefoil.

E. The trefoil is the simplest knot that is nontrivial.

PROBLEM 2. Choose the variation that is least choppy.

A. Following the publication of the first paper in this area [OS04], indeed, two counterexamples have been found (for $k = 2$ and $k = 5$).

B. Indeed, two counterexamples have been found (for $k = 2$ and $k = 5$), following the publication of the first paper in this area [OS04].

C. Indeed, following the publication of the first paper in this area [OS04], two counterexamples have been found (for $k = 2$ and $k = 5$).

D. Two counterexamples have been found, indeed, for $k = 2$ and $k = 5$, following the publication of the first paper in this area [OS04].

PROBLEM 3. In which of the following sentences would it be acceptable to say *I*? The context is given in italics.

A. I would like to thank my father and brother for their support during my PhD. (*In the acknowledgement section of a thesis.*)

B. Let me first show how the theorem can be applied to the case of the 3-torus, then I will prove the theorem. (*In the body of a thesis.*)

C. I spent a semester doing an internship with a language-processing firm, where I learned how to analyse texts using Python. (*Personal statement in a PhD application.*)

PROBLEM 4. In each of the following sentence fragments underline the hedge words or phrases that are being used as nuanced verbal shortcuts for complexity.

A. Through various algebraic arguments, we can show…

B. The proof is somewhat different than…

C. Keeping track of the indices is quite tricky…

D. For technical reasons, we often resort to…

LECTURE 8

How to Write a Good Personal Statement

Whether you are applying for a Master's or for a PhD, the personal statement is your chance to present a coherent picture of your traits and abilities, and convey a bit about your personality. In this lecture, we discuss general best practices for writing such a personal statement. We discuss the purpose of the statement (Section 8.2), the information that should be included (Section 8.3), and finally some writing tips (Section 8.4).

However, what these notes will not do is teach you how to write your personal statement from scratch, sentence by sentence, or tell you what certain universities or people expect to hear from you.

REMARK 9.1. For the remainder of this lecture we say *Statement*, with a capital *S*, to mean some or all of the various personal prose communications that you may be asked to attach to an application. Depending on the country or university, this Statement comes under the heading of personal statement, cover letter, statement of purpose, statement of objectives, and so on.

8.1 Disclaimer

The following information is based on the experiences of the author (as an applicant himself) and on his limited experience of the second author in selecting PhD and postdoc students of his own. Therefore, even more so than elsewhere in these notes, what you will read are mere guidelines and suggestions for best practices. All students should conduct thorough research of their own into the requirements of the universities they are applying to. (See Section 8.3.4 for a slightly more detailed discussion.)

If you are not a mathematician, and will be applying to do a different science degree, please substitute in your subject every time these notes say "maths". Also, the less your subject is related to pure mathematics, the more these notes are likely to err in terms of what your application should look like. (Though some aspects of our advice remain universal: write good English is one of them.)

8.2 The purpose of the Statement

Suppose you were not required to submit a Statement, just your academic record and your references. In that case, a member of the application committee could still learn a lot about you: your grades reflect your ability to learn for exams and perform under that kind of pressure; your CV lists all the relevant academic and extracurricular milestones; your references describe the impression you give off as a student and comment on your talent. What is missing is your own voice, arguing your case.

Put differently, the Statement is the only place in your application where you are allowed to make two essential arguments that are not explicit elsewhere:

I. Why you think you *would* make a good Master's or PhD student (or postdoc).

II. Why you think you *should* be a taken on as a Master's or PhD student (or postdoc) at a particular institution or by a particular researcher.

Making these two arguments is the purpose of your Statement: the first establishes your motivation and relevant background, the second explains why you would be a good fit for the particular position. The two arguments are not always disjoint, but everything else—the formatting, the phrasing, the additional information you add—is there to pad out the text and prove you can string together a few decent sentences. Nonetheless, you have to get the whole package right (not just arguments I. and II.), or, at a minimum, you must not get any of it wrong! Here is why.

8.2.1 Getting all the elements right

First impressions matter, and two things will ruin the first impression of anyone reading your Statement: getting the form of address wrong and making English language errors. Next, the reader will notice immediately whether the Statement contains relevant information (I. and II.) and whether it is structured in the standard way (introduction, body, conclusion, and not a haphazard collection of thoughts). If all of these hurdles are cleared satisfactorily, your Statement will be given due consideration. At that point, you have to make sure you are not using overblown language and are not massaging half-truths to support your case, because a careful perusal will reveal any hollow boasts. Finally, if the rest of your Statement is well-written, the reader may appreciate a light touch of humour or a rhetorical flourish that shows a tad more of your non-academic personality. We proceed to discuss all of these elements.

8.3 Relevant information

Many Statements include the wrong kind of information, such as:

- too much personal history, which is either outdated (stories from childhood) or irrelevant to the subject (hobbies);
- too few appropriate subject-specific detail: a lot is said about maths in general, and not much about what makes the applicant a worthy mathematician;
- too few precise references to the university of choice (e.g. usually people say it is prestigious and in a good geographic location), which is a bare, almost embarrassing minimum that distinguishes you in no way from dozens of other applicants.

In this section we warn against such mistakes and offer what we think is a healthier balance of information.

8.3.1 Personal history in the Introduction

Especially for American Statements, it is often recommended that you try to engage the reader in the introduction by recounting a memorable personal anecdote. You do not have to include an anecdote if there are none. However, if you decide to include an anecdote, you do yourself no favour by starting with a memory of how "I fell in love with maths in fifth grade because I noticed I was good at geometry" or how "I became passionate about pursuing a degree in engineering after I played with origami as a child".

Let us see how you could improve on the first sentence to make it a bit more relevant.

To begin with, try to include a maths fact or some terminology.

> I fell in love with maths in fifth grade when the teacher stated you could not square a circle with compass and straightedge in a finite number of steps.

Secondly, try to avoid general words like *love*; instead, relate a concrete incident.

> When my fifth grade teacher stated you could not square a circle with compass and straightedge in a finite number of steps, I spent the whole evening trying to do just that, convinced I could disprove her.

This is already much better. What is missing now is the relevance to your current aspirations (Galois Theory, for example).

> When my fifth grade teacher stated you could not square a circle with compass and straightedge in a finite number of steps, I spent the whole evening trying to do just that, convinced I could disprove her. When I failed, I wanted to know why. Eventually, I went on to learn about π, transcendental and algebraic numbers, and polynomials, and finally Galois Theory at university, but by then my interest in the subject was firmly established and had evolved far beyond the ancient problem from geometry that started me on this path.

This would be a solid *first draft* of an introductory anecdote. Depending on the remainder of your introduction and the rest of your Statement, you would remove some of the elements. Regardless, you would need to edit these sentences to about half the current length, because your Statement should be chiefly about your most recent work, not your enthusiasm as a child.

See Section 3.2.4 for more about *opinion* words, words that require justification or that are simply overused.

> Only ever include relevant incidents, and be specific about how they are relevant. Use concrete (but not overly technical) maths terminology whenever possible; avoid opinion words.

8.3.2 Personal history elsewhere

Personal history details that occur in the Statement fall into two groups:

i) facts mentioned in the records,

ii) facts not mentioned in the records.

The records encompass any other formal, factual documents whose contents you are familiar with; most commonly, these are your academic record and your CV (unless you have also read your recommendations).

Facts mentioned in the records. Your CV will contain dates, names, and GPAs. When drafting your Statement, you can assume the reader is familiar with all of that information or has it at hand. Therefore, you do not need to repeat yourself. On the other hand, you can and should freely reference facts. Here are some examples:

- BAD: I finished my Bachelor's degree in the Summer of 2017.
  (*The whole sentence is redundant; it should be obvious from the CV. It becomes relevant only if there is a some greater point you are about to make. For example, that you took a gap year to do charity work, and it was important that this gap year was 2017–2018 because that is when a particular disaster happened—a socio-historical fact not mentioned in the CV. This is an extreme example, but you should need an extraordinary reason for explicit repetition.*)
- GOOD: While completing my Bachelor's degree, I...
  (*Appropriate referencing of information known from the CV.*)
- BAD: During my internship (May–June 2016 at Zrootech), I learned how to...
  (*The parenthetical information belongs in the CV.*)
- GOOD: The conferences I attended last summer gave me an opportunity to meet various researchers in my field, and talk to them about...
  (*The CV should include information about the conferences, locations, dates, topics, so this is an appropriate reference that can be followed up via the CV.*)

Whilst you should not repeat yourself, you should also not tend to the other extreme of avoiding all references to facts from your CV. That would make writing the Statement difficult, and it would also mean that you miss out on an opportunity to offer supporting, if roundabout, evidences for arguments I. and II. Specifically, any details that you mention in the Statement that are also mentioned elsewhere are viewed as one of the following:

- A "highlight": if you won an award, had a particularly successful thesis, or achieved anything that is outstanding in recent years (not in kindergarten!), then you are entitled to talk about this event in the Statement and say how it has influenced you. What is more, you are expected to do so, and it would be strange if you chose not to.

- An "explanation": if there were any unexplained gaps in your CV, or you took a non-standard route towards accomplishing your degree, or you have a medical reason why you do badly on exams, then you are expected to help the reader understand the circumstances and how they have influenced you. As a rule do not complain or seek sympathy, instead always talk about the lessons you took away from difficult experiences.

Events that make you stand out will fit in either of those categories, highlight or explanation, or even might be considered as both. For example, a six-month internship with a railroad company looks like it needs *explaining* if you are applying to do a PhD in Statistics, until you say that you were learning on the job about route and time-table optimisation, in which case it could become a highlight.

> Use the Statement to "highlight" or "explain" any facts on your CV that make you stand out, thus making them even more memorable. Focus on what you have learned or gained from experiences, rather than repeating facts or becoming emotional (bragging, complaining etc).

Facts not mentioned in the records. Hard facts (dates, names, numbers) really should be mentioned in the records. As we saw Section 8.3.1 and in the examples of the previous section, any other personal history that you end up relating beyond that will be anecdotal. Your anecdotes may or may not be verifiable. Naturally, if you are relating a conversation with professor or a public experience like a presentation, then you have to be careful not to state obvious falsehoods about time, place, and content. If you are talking about your internal motivation and gain, then you should portray yourself as best possible, with a caveat.

Caveat: be careful of exaggeration in general, and cultural sensibilities in particular. Some assertions might come across as appropriately bold in an American application, but might seem like boasting on a British application.

- I am especially keen on your program because it would allow me to interact with and learn from top specialists in my field, specifically Professor Juniper, the Fields medallist who has recently taken up a post at your university.
  (*More forceful, forthright, confident. To British ears might sound presumptuous that you think you would get to* <u>*interact*</u> *with such an esteemed member of staff.*)

- I would especially appreciate the opportunity to attend the seminar series given by your Fields medallist, as I am interested in one the research areas she developed. Namely, I would...
  (*More subdued, realistic, humble, and in line with British sensibilities. Might sound unambitious to American ears.*)

Do you notice the difference in tone? Neither is better or worse, and you should not try to affect a tone foreign to you, but be aware that small word choices, viewed cumulatively, will influence the way you come across. See Section 8.4 for more tips on form of address and tone.

All important facts should be mentioned in the academic record or CV. Any other anecdotal and personal information should showcase your experiences in the best possible light, honestly and gracefully.

8.3.3 Making argument I: subject-specific detail

The first argument (and more important one in a lot of ways) is for why you think you *would* make a good Master's or PhD student. Traditionally this argument is made by directly answering the following questions in your Statement.

i) General motivation: Why do you want to do an advanced degree?
This is a natural, open-ended question, but you are not expected to give a creative answer. It will suffice to say that you have long-term interest in a particular subject, or wish to gain insight into the subject before going into industry or continuing a PhD.

ii) Skill-specific question: What research skills or experience do you have that make you think you would be a good Master's or PhD student?
For this you mention your most relevant skills: specialised courses, reading courses, projects, seminars, programming experience from an internship, other practical experiences. This is where you are aiming to "highlight" facts from the CV.

iii) Personal motivation: Why have you chosen your subject area and not some other? What interests you within it?
This is your opportunity to deliver a an insight into your mathematical thinking that appears nowhere else, and that shows your interest is not merely superficial. For example:

> I particularly enjoy investigating which structures an invariant can "see" and applying its properties to distinguish two otherwise similar spaces, ranging from elementary examples: CP^2 and $S^2 \wedge S^4$ are not homeomorphic as their cohomology ring structure is different, to more complex ones: there exist diffeomorphic Calabi-Yau 3-folds with different quantum cohomology rings.

Remark 9.2. In answering iii) you should bear in mind your Ideal Reader, as discussed in 7.1.4. In other words, depending on who you judge will be reading your Statement, you should adjust your language. If you are addressing a professor in your future field, any specialised technical terms will be understood and appreciated; if you are addressing a member of a general admission panel, you should stick to more general maths terms.

You should plan to spend between one and three sentences answering these questions somewhere within the structure of your Statement.

Make yourself memorable by including a sentence on two on the mathematical content that interests you most.

8.3.4 Making argument II: application-specific detail

You have to convince whoever is reading your application that you are applying to them or their department, and not that you copy-pasted the text for some other university. There is a good reason for this: no advanced program wants to accept people who do not actually want to attend. In other words, as much as you are worried whether a position will suit you, the university offering that position is worried about the same thing. Because, once accepted, if you are unhappy, you are likely to be unsuccessful and that will also likely make the university look bad. In forcing you to write a few relevant sentences about why you think they should admit you, universities ensure that you have at least done the basic research about their degree program and that you are convinced it is a decent fit. Universities always have too many applicants and would rather that people self-selected for good fit (academically and otherwise) before applying.

Thus, in your statement you are expected to answer these basic questions (when applicable):

i) **When applying to a person:** What drew you to this particular academic? Which of their research do you find most interesting and why?
Here it is good to list a specific paper or book they have written and you have at least glanced at, or if you took a course or seminar with them, even better, comment on what you enjoyed and why. Obviously, if you have met in person, even briefly, then remind them of this if the encounter was at all positive.

ii) **When applying to a department (but also to a person):** What drew you to this particular department or university? What research conducted in the department do you find interesting?
If you can, avoid more than the briefest mention of general ranking or prestige, or of location. The second question suggests the kind of answer you are expected to give: mention research groups at the department, any facilities that they provide and that you need for your work, specific courses that you need for your degree that may not be offered elsewhere, etc.

iii) **When applying to a taught Master's program:** If the program envisions a particular curriculum, what makes you suited to that study that curriculum? If there is flexibility, which areas do you intend to focus on?
This is where you can "highlight" facts from your CV and say how this background has prepared you well for the curriculum, or what areas you will build on.

iv) **When applying for any research degree:** How do you see yourself contributing to the research or to the research life at the department?
This is an open-ended question where you are not expected to give a particularly creative answer. Most well-meaning answers will do: contributing to a particular research group, hoping to develop a skill set which will complement those of others, working closely with other researchers in a similar area.

This is a bit of a chicken-and-egg cycle: to answer these questions you have to know where you are applying, but in order to know where you are applying, you need to have answered these questions at least partially (in your head). These lecture notes assume that you already have a number of names and institutions in mind, either due to repeating this cycle (and narrowing down your options) or due to recommendations from senior colleagues and professors that seeded your search. Therefore, it hopefully remains just to crystallise these answers and commit them to paper.

Before writing each individual application, remind yourself of why you are applying to a particular program. If it was due to a recommendation, that might be worth including sometimes, but most of the time, it is hardly sufficient (or advisable) to say: *Professor Juniper says that I should apply to you.* Go and find out why that professor recommended this program, and write down those reasons. In particular (when applicable):

- Find out more about the person you are applying to. Look at their CV, at their official department page, at a list of their research, at the abstract of those research papers that seem most relevant to you.
- Find out more about the department you are applying to. If it is a taught Master's program, look at the details of the coursework.
- Find out which research groups the department has, what their specialities are, whether you would fit in and how. It is good to see some famous names, but it is more important to see the speciality areas that you might be looking to join.
- Sure, learn more about the prestige of the institution and its location. But even though these factors may play a huge role in your own decision process, do not emphasise them in the statement because the institution is well-aware of its own prestige and the attractiveness of its location (as are all the other candidates), so emphasising this will not gain you anything.

> Find out more about the person or department you are applying to, then give a specific reason as to why you think you would be a good fit for the position they are offering. The more precise your answer, the more it will be obvious that you care about being admitted.

8.4 Writing tips

8.4.1 Get the name right

If you are applying to a professor in person, then make sure to get their name right as well as any titles.

If you are applying to a department and no name is given, or an impersonal online application form then it may acceptable to use the standard impersonal form of address: *To Whom It May Concern.* Though, whenever you can, try to find out whether there is someone you can address it to (perhaps the Head of Department).

8.4.2 Follow standard structure

The letter follows the standard structure of: Introduction, Body, Conclusion. The precise content is difficult to define for all the different kinds of Statements. For example a paid, taught Master's program in the UK is less likely to expect introductory anecdotes about why you got into maths, but an American university might appreciate you hooking them with an imaginative opening. Look up sample letters for the country you are applying to (America, UK, and others) and the type of program you are interested in (Master's or PhD), then follow the guidelines outlined in these notes to ensure you address all the things that are expected of you.

8.4.3 Do not use bombastic language or cliches

Throughout this course we have urged you to rephrase sentences to remove redundant words, as well as, when appropriate, to remove opinion words (Section 3.2.4). In your Statement opinion words are welcome, though whenever you can back them up with concrete reasons.

> I find knot theory *interesting*, especially the idea of refining Floer theoretic methods to investigate the properties of knots.

The second part of the sentence does not actually say why you find knot theory interesting, but it does indicate that you have been thinking about the subject (itself an indication of interest).

That said, you should scrutinise your use of adverbs: *always, best, never, well,* and all those ending with *ly*. You should have no need for *very*, especially in phrases such as *~~very~~ interested, ~~very~~ excited, ~~very~~ enthusiastic, ~~very~~ good, ~~very~~ helpful*. The same is true of *really*. In the same vein, avoid creative words that may not use superlatives, but are meant to sound like superlatives. Also avoid cliches and colloquialisms. Here is a sample of what not to write under any circumstances:

- I am bursting with enthusiasm.
  (*This metaphorical meaning of bursting does not belong in formal writing.*)
- I am highly motivated.
  (*The adverb should be remove, but then so should motivated unless you are following it up with a good reason.*)
- I love my subject.
  (*This is understood and expected.*)
- The realisation that I wanted to do a PhD blew my away.
  (*The phrase blew me away is informal and does not belong in your Statement.*)
- My childhood dream, my dream, my utmost desire.
  (*You are too old to refer to dreams and desires in this way.*)
- My greatest ambition, I am ambitious, my ambition has paid off so far.
  (*This sounds either like bragging or like you are running for president and hoping to build orphanages. It is best avoided.*)

- Since the age of five I have been fascinating with science.
  (*Whilst this may be true, the statement is too general to be helpful, and must be either deleted or modified as illustrated in Section 8.3.1.*)

8.4.4 Be truthful.

This may sound obvious, but it is surprising how easily one can get carried away with embellishments and small alterations that seemingly slant the Statement in your favour. Typically this involves one of the following scenarios:

i) You decide to inflate a (programming or language) skill. For example, you might write: *During my internship, I learned to use MATLAB*, actually you opened MATLAB twice during that time. Commonly, this happens when you think no one can actually check whether your claim is true or not until after they have decided to hire you.

ii) You decide to unduly highlight a subject area. For example: *In my third year I attended an advanced seminar on Kirby Calculus,* may sound impressive, until it becomes apparent that you attended it by sitting in the back, staring blankly at the board, or playing with your phone. This is kind of truth slippage occurs most commonly when you need a list of two items to look longer, so you add a few more things you attended, or a few more things that would look good if they were true.

iii) You decide to get help with writing advanced maths in your Statement. You have correctly identified that coming up with an advanced answer to iii) in Section 8.3.3 is a good idea, so you ask an older friend to supply you with a maths insight. Alternatively, you open a textbook or paper, select an important-looking result, and regurgitate it in your statement.

Depending on your sophistication and on how far away from the truth you have strayed, each of these scenarios will be more or less detectable. Your Ideal Reader has probably seen hundreds of such applications, and can detect a *lie* (let us call it what it is) without even thinking about it.

It is best if you self-police, and detect such scenarios in your own Statement before sending it off. The question you should always be asking yourself is as follows:

> If I am asked to demonstrate a skill, talk about an experience, or discuss technical knowledge would a charitable interviewer buy my story?

Here *charitable* means, you imagine a friendly interviewer, who is willing to take into account that you are nervous or that you may have forgotten some details. In the examples above, no matter how charitable the interviewer the student who boasted of knowing MATLAB, Kirby Calculus, and an advanced result would have been busted.

> There are shades of truth and shades of lies. Stay as close to the truth as you can; ask yourself whether you would be able to look an interviewer in the eye, if challenged, and *plausibly* justify every sentence in your Statement.

8.4.5 Use a professional tone

Reread Sections 7.2.1 and 7.2.2 on tone and tone meets pitch with an eye towards writing the Statement.

8.4.6 Humour

Everyone likes a joke or two, but the Statement is hardly the place to try to be funny. Avoid humour, puns, and anecdotes that are supposed to be amusing. That said, if you have met the person you are applying to or have seen them speak and are aware of their sense of humour, *and* you are confident you can say something that would fit their sense of humour, then you could include an appropriate reference or humorous comment. Though, we cannot emphasise enough how cautious you should be when doing so.

> Do not use humour in your Statement unless you are certain it will be appreciated.

8.4.7 Editing and proofreading

When it comes to editing your draft, think of your Statement as if it were a proof: every single line needs to be justified either directly or indirectly (as shown in the first example of Section 8.4.3). Not only should every line be justified, but it must be specific. A way to check for specificity is as follows: Take your Statement and replace the word *mathematics* every time it appears with the words *molecular biology* or *art*. If there is a single sentence in your Statement that makes sense after this substitution, then you should rework that sentence—it is not specific enough!

- Ask a friend to take a look at the Statement and tell you what they think (does it leave a good impression, does that impression match who you are, are there any glaring errors, etc).
- Google phrases that you are unsure of.
- Check the spelling and grammar.
- Read the Statement aloud.
- Leave the text to sit for a few days before looking at it again.
- Be careful when copy-pasting bits into other statements—it's easy to paste in a detail which obviously does not belong.

> Take the time to get the details right.

Problem Sheet 8 (Non-examinable)

PROBLEM 1. Write an excellent Statement and get into a program of your choosing.

LECTURE 9

Notation and Conventions

Successful communication within a scientific subject depends on all participants following a set of subject-specific conventions, and in particular, it depends on everyone using and extending notation in a standard way.

In professional publishing, editorial style comprises a set of in-house rules that are independent of the author: they ensure consistent linguistic conventions are used throughout all of their content ensuring visual uniformity. These rules include spelling, punctuation, and capitalisation conventions, abbreviations, choice of font, use of italics and bold type, layout of material, margin size, and so on.

For theses, preprints, and less guided pieces of writing, it is up to the author (and adviser, if applicable) to decide these editorial matters in line with university guidelines or accepted standards. All of what is covered in this course, and specifically what is covered in Lecture 5 (on punctuation) and today's lecture (on notation and conventions), should prepare you to make those decisions yourself or understand why they are made for you.

9.1 Choosing Symbols

Notation takes mathematical objects (their classes, their instances) and gives them shorthand names, which can then be usefully manipulated to make meaningful statements about the mathematical objects. In short: notation is a system of symbolical labels.

Does it matter what something is called? Once you define

$$\mathfrak{X}^{i\{C\}}_{6h_\alpha} \colon \mathbb{R} \to \mathbb{R}, \quad \mathfrak{X}^{i\{C\}}_{6h_\alpha}(x) := x^2,$$

does it matter that the decoration was superfluous and that you could have called this function $\mathfrak{X}$? Or indeed, that you could have simply called it f? Philosophically speaking, no, it does not matter, because mathematical truths are independent of labels. Practically speaking, good notation is crucial for two reasons: it allows you to write down a proof that convinces *you* a theorem is true, and it allows you to write down a proof that convinces *others* your theorem is true.

Points to remember when choosing notation:

i) Define a symbol before you use it.

ii) Do not introduce unnecessary symbols.

iii) Use conventional notation.

iv) Be wary of unintentionally repeated symbols or symbols that clash.

v) Be consistent.

We discussed i) and ii) in earlier lectures. Below, we discuss the remaining points.

9.1.1 Handwritten versus typed maths

It is almost impossible to distinguish italicised letters from upright letters in a handwritten text, whereas it is easy to distinguish f from f in a typeset text such as this one. Furthermore, even though it may appear so, typeset maths is not written using italicised symbols in the standard way (the way it would in a word-processing editor, in markdown, in html, etc). Compare

f(x),

written in italics, with

$$f(x),$$

written in so-called *math mode* in LaTeX. Aside from the parenthesis being upright in math mode, the spacing between letters (called *kerning*) is arranged to make the symbols more obviously part of a mathematical expression.

Different fonts and symbols that regularly appear in textbooks or papers may also be difficult to render in handwritten notes. For example, you would likely struggle to tell apart some or all of these G's in a lecture, especially if seated further away:

$$G, \quad \mathcal{G}, \quad \mathfrak{G}, \quad \mathscr{G}.$$

Some pieces of notation are more susceptible to this problem than others, e.g. vectors in $\mathbb{R}^n$. There are different ways of denoting vectors:

i) Using the regular typeface and relying on other information to convey that this is a vector: v.

ii) Using the bold typeface: $\mathbf{v}$.

iii) Using a line above or underneath the letter: $\overline{v}$ or $\underline{v}$.

iv) Using an arrow above the letter: $\vec{v}$.

Some of these ways are better than others. The bold typeface may be removed by some journals and is typically difficult to render in handwriting, so these lecture notes discourage its use. The regular typeface is preferred in advanced mathematics, because a vector is an element of a vector space, and we do not use special notation for elements of other vector spaces (those of functions, matrices, etc). That said, if special notation is needed for pedagogical reasons, use options iii) or iv).

9.1.2 The symbols and fonts to choose from

For a short equation or theorem you only need a few symbols, but for longer texts, choosing notation that is compatible and unrepetitive becomes tricky. Depending on how you look at it, there appears to be either a fairly limited number of available letters from the Roman and Greek alphabets, or a fairly large number that you cannot imagine ever exhausting. Regardless, you should be aware of your choices and how you can combine them. Here are the capital letters side by side; note that some repeat.

Roman: $A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z.$
Greek: $A, B, \Gamma, \Delta, E, Z, H, \Theta, I, K, \Lambda, M, N, \Xi, O, \Pi, P, \Sigma, T, \Upsilon, \Phi, X, \Psi, \Omega.$

If you are new to LaTeX, you might be curious what the beginning of that second line would look like in an editor before compilation:

```
A, B, \Gamma, \Delta, E, Z, H, \Theta, I, K, \Lambda, M, N, \Xi, O,
```

The names of the letters that do not exist in English are written out in full, and start with a capital letter after the backslash. Lowercase letters are written the same way, except their names begin with lowercase letters. So `\Gamma` becomes `\gamma`, giving Γ and γ, respectively. (Do not worry about the syntax yet, this is merely an illustration that will help the explanations to come.)

As you see, recalling, distinguishing, and spelling the names of Greek letters is crucial. In order of appearance above, the letters are: alpha, beta, gamma, delta, epsilon, zeta, eta, theta, iota, kappa, lambda, mu, nu, xi, omicron (but you just write o!), pi, rho, sigma, tau, upsilon, phi, chi, psi, omega.

Next, note that four of the Greek letters have *var-* alternatives given in parenthesis. In LaTeX these are written as `\varepsilon`, `\vartheta`, `\varrho`, `\varphi`, respectively. The prefix stands for *variation*, which are inherited from the variations in Greek script.

Roman: $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z.$
Greek: $\alpha, \beta, \gamma, \delta, \epsilon(\varepsilon), \zeta, \eta, \theta(\vartheta), \iota, \kappa, \lambda, \mu, \nu, \xi, o, \pi, \rho(\varrho), \sigma, \tau, \upsilon, \phi(\varphi), \chi, \psi, \omega.$

There are three more *var-* letters that are rarely used: `\varkappa`, `\varpi`, and `\varsigma` corresponding to $\varkappa, \varpi$, and ς.

To gain additional variation in letters, mathematicians turn to different fonts. The most common ones are `\mathcal`, `\mathfrak`, `\mathbb` and are usually applied to capital Roman letters. By *apply*, we mean that in LaTeX you would write `\mathbb{R}` to get $\mathbb{R}$. Another font that you might encounter is `\mathscr` (though, to use it, you need to add a package called `mathrsfs` in the preamble).

cal: $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}.$
frak: $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \mathfrak{I}, \mathfrak{J}, \mathfrak{K}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N}, \mathfrak{O}, \mathfrak{P}, \mathfrak{Q}, \mathfrak{R}, \mathfrak{S}, \mathfrak{T}, \mathfrak{U}, \mathfrak{V}, \mathfrak{W}, \mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}.$
bb: $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{H}, \mathbb{I}, \mathbb{J}, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{N}, \mathbb{O}, \mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S}, \mathbb{T}, \mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{X}, \mathbb{Y}, \mathbb{Z}.$
scr: $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{I}, \mathscr{J}, \mathscr{K}, \mathscr{L}, \mathscr{M}, \mathscr{N}, \mathscr{O}, \mathscr{P}, \mathscr{Q}, \mathscr{R}, \mathscr{S}, \mathscr{T}, \mathscr{U}, \ldots.$

Occasionally, other letters might show up, like aleph, the first letter of the Hebrew alphabet, shows up in the cardinality of the natural numbers $\aleph_0$, which is written `\aleph_0`. Lowercase letters and numbers might appear in another font, like $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ to denote Lie algebras, or $\mathbb{1}$ to denote the identity or the indicator function of a set. Moreover, there are certain instances when a mathematical term calls for upright letters, rather than italicised, in which case it is written using `\operatorname`. For example, in the expression $\operatorname{Hom}(A, B)$, the Hom functor from category theory is written using upright letters:

```
\operatorname{Hom}(A,B).
```

Writing simply `Hom(A,B)` in math mode results in the awful-looking $Hom(A, B)$.

In LaTeX there are a set of preexisting commands. For example, `\sin` prints the sine function in $\sin\alpha$, as compared to just writing `sin \alpha` that gives $sin\alpha$. These preexisting commands are handy and you should use them whenever you can. However, there will be specific operators or strings that you repeatedly use in math mode and that you may not want to write our every time. In this case, you can chose to define your own commands at the start of the document. For example,

```
\DeclareMathOperator{hom}{Hom_{\mathsf{C}}},
```

allows you to type `\hom` and get $\operatorname{Hom}_{\mathsf{C}}$.

Finally, your displayed equations may include English words. As displayed equations are an extended version of math mode, you need to tell LaTeX to treat those words as ordinary text. For this you would use `\text`. For example, the following expression

$$X(t) := \{x \in M \mid f(x,t) = 0, \text{ for some } t\}$$

is written as

```
X(t):=\{x \in M \mid f(x,t)=0, \text{ for some } t\}
```

9.1.3 General notational conventions in English

When it comes to choosing the symbols yourself, there are a few guidelines that you already probably know intuitively.

Integers are usually represented by symbols from the middle of the Roman alphabet, such as i, j, k, m, n, especially if they are used as subscripts or superscripts. **Prime numbers** are p and q.

The lowercase o is rarely used because it resembles the digit 0. Also o and O are used for the "little-o" and "big-oh" notation $o(\log n)$ and $O(\log n)$ etc.

Real numbers are x and y. If the real number represents time, then t is often used. (In general pretty much anything works for a real number, as long as it does not clash with other notation.)

Complex numbers usually take the letters at the end of the Roman alphabet, either z or w as a first choice, and $z = x + iy$ is the typical notation for a number in the complex plane, with x and y the real and imaginary parts. Though, in polar coordinates the standard notation is $z = \rho e^{i\theta}$ or $re^{i\theta}$.

Writing the symbol $i = \sqrt{-1}$ can be problematic. The standard option is i, which is inconvenient if located next to other instances of the symbol, such as in the following examples:

Symbol	Size
ε	small
δ	small
n	medium
N	large
$\aleph_0$	very large
$\hbar$	very small

Table 8.1: Certain symbols and letters are usually thought to be "large" or "small". Most of these are implicitly assumed positive. (Shortest math joke ever: Let $\varepsilon < 0$.)

- Let i and j be positive integers, now consider the complex number $i + ij$.
- Let $\{a_i \mid i \in I\}$ denote a set of complex numbers. Now define $b_i := ia_i$.

There are various ways round this: you can write $\sqrt{-1}$ or you can use a different font $\mathbb{i}$, or, better still, you can simply ensure nearby symbols are not incompatible (see below for more about repeating and clashing notation).

Functions are typically named f, and then g or h. However, in some cases the uppercase letters are preferred, e.g. $F(x, y) = (f_1(x, y), f_2(x, y))$. Labels for the standard variables correspond to those of unknowns. Sometimes functions take Greek letters. Famous examples include the Riemann zeta function $\zeta(z)$ and the Gamma function $\Gamma(n)$.

Sets are represented by uppercase letters, with typical generic sets denoted by A, B, C and X, Y, Z. Whenever possible, the label for the set reflects the nature or labelling conventions of its elements. So it makes sense to label a sequence of functions as $\mathcal{F}$, a set of point $\{p_1, \ldots p_k\}$ as $\mathcal{P}$, a set $\{a, b, c\}$ as A, and so on.

Other **composite objects**, such as groups, functions, matrices, and spaces, are typically denoted by capital letters that reflect their names. So a group or a graph is G or Γ; a manifold is M, a moduli space is $\mathcal{M}$. If there are more than one element of the same kind, adjacent letters are used or indices or primes. So we speak of manifolds M and N, of groups G and H, of spaces X_1, X_2, X_3 and Y, Y'.

Similarly, symbols for **modified objects** are typically chosen to be closely related to the symbols of the objects that were modified. For example, one could use $\mathbb{Z}^+$ to denote the positive integers[1], and $\mathbb{F}_p$ denotes the prime field obtained from the integers modulo a prime number p. Usual tricks for distinguishing derived objects include (but are not limited to): indices x_i, prime x' and double prime x'', asterisk x^*, overline $\overline{x}$ and underline $\underline{x}$, and tilde $\tilde{x}$.

As a general rule, do not combine similar decorations. If x is going to get a tilde, do not give it a hat as well: writing $\hat{\tilde{x}}$ is bad form.

9.1.4 Conventions in specific fields

In any given field of mathematics or physics there are specific notational conventions, and you should adhere to them.

There is no way to "guess" what the correct conventions are—you learn by osmosis (which means over time, by reading books and papers).

[1] The authors of this text prefer $\mathbb{N}$ though!

Mathematical object	Standard symbol
symplectic form	ω
contact form	α
contact distribution	ξ
Riemannian metric	g
almost complex structure	J
point in a cotangent bundle	(q, p)
Hamiltonian function	H

Table 8.2: Standard symbols in symplectic and contact geometry.

9.1.5 Unfortunate choices

So far we have seen that there are many symbols to choose from but that convention demands you use only certain recognisable combination. In a sense, it is a fine line to walk: you must conform to expectations, but you may also have to reach for unused letters in longer papers. Veering off that fine line results in one of the following issues:

- **Repetition:** At the beginning of Section 1, you define a function f. At the beginning of Section 2, you define another function f, which is alright because it is used to prove a completely disjoint set of theorems than those in Section 1. However, in Section 3 you wish to bring the results of the previous two sections together, and you suddenly have to deal with two completely different functions both called f. Alternatively, you may have called a vector space V, but also be dealing with open sets U and V. Or have a field F, and the fibre F of a fibre bundle. This issue is more common than you think and it is unforgivable. Always relabel your objects!

- **Clashing symbols:** You are working with $\mathbb{R}^n$ and $\mathbb{R}^m$. Later you define manifolds M and N. Later still, you realise that because of the way you have set up your notation, M is of dimension n and N is of dimension m. Whenever possible (and it may not always be possible) avoid such clashes.

- **Exotic variations:** You are running out of names for your sets, you have used up all the symbols, so you reach for another font, and you end up talking about an element in the intersection:

  $$a \in \mathcal{A} \cap A \cap \mathscr{A}.$$

  In general it is better to decorate the base symbol than use different fonts to label instances of the same kind of object.

9.1.6 Define all labels

You will probably understand the formula $E = mc^2$ (that E is energy, m is mass, c is the speed of light in a vacuum). However, you should always specify exactly what you mean if there is any danger of confusion.

9.1.7 Be consistent

Do not alter your notation midway through any unit of text that relies on that notation.[2]

[2]This is equivalent to writing a novel and midway through changing the names of some of the characters.

LECTURE 10

How to Write a Good Thesis

Writing a thesis is daunting, not least because it requires that you to first understand complex maths and then prove your understanding through a coherent, lengthy exposition that must withstand close scrutiny. Unlike oral exams where you can muddle through with hints and hand-waving, or written exams where set questions expect set answers, in a thesis you are on your own. Even a Bachelor's thesis that requires the least original thought can be written in multiple ways using multiple approaches that depend on the topic and your background and personal preferences. Nevertheless, most papers in mathematics closely follow structuring conventions—the ordering and presentation of ideas—and theses are no exception. Once you know what is expected, doing the actual work will seem less daunting. Moreover, once you have completed your first thesis, writing long pieces of maths will be comparatively easy.

In this lecture we walk you through the process of writing a thesis: in Section 10.1, we discuss the preparation phase; in Section 10.2 we go over the conventions for writing local elements such as Theorems and Roadmaps; in Section 10.3, we give the global layout of a thesis; and finally, in Section 10.4, we go over some hints for revision.[1]

Remark 10.1. From now on, we say *Thesis* with a capital T to mean any of the three types of theses: Bachelor's, Master's, and PhD. If we need to refer to an individual type, we use the shortened form *BSc, MSc,* and *PhD thesis.* Furthermore, the words *a good Thesis* are understood to mean *a Thesis most students and their advisers are happy with* and *a Thesis which will get you where you want to go in life.* We have chosen this general expression to embody the ideal result you would produce after having done the hard maths and written up your work using the advice offered in these Lecture Notes.

Remark 10.2. Semester projects are sufficiently similar to BSc theses that we do not discuss them separately.

Remark 10.3. Even though this lecture is geared towards writing Theses, much of our advice is applicable to writing papers in general. Whenever appropriate, we remark on the relevant differences between Theses and papers.

10.1 The preparatory work

To have something to write, you have to have something to say. But that something has to be of the correct level of detail and understanding. So the first step towards writing a good Thesis is doing the correct kind of maths (reading, rewriting proofs,

[1] This paragraph is an example of a *Roadmap*. See Section 10.2.2 for details.

researching new proofs, proving new theorems). Knowing what to do, when to insist and when to desist in trying to understand or explain something is an important skill. And it is not as obvious a skill as you might think.

Every degree program that ends in a Thesis, automatically makes that Thesis somewhat of a means to an end: rather than to engage with the underlying maths for the sake of the maths, the chief goal of the student inadvertently becomes to write a good Thesis. This is more true of MSc than of BSc theses, and it is most true of PhD theses: the better your PhD thesis, the more papers you can generate from it, the more talks you can give, the more interest you can garner for your work, the better your future prospects. There is nothing mysterious about the situation; it is similar to the one you face at every exam: to pass the exam, you do not have to know all the material, you just have to know the material you are examined on and know it up to a sufficient (often rather incomplete) degree. So to write a good Thesis, is it sufficient to present only the material you understand well and ignore the rest? The answer is both yes and no, and striking the wrong balance between the two extreme answers may have some unfortunate consequences.

10.1.1 One extreme: understanding everything.

You may think that the only way to write a Thesis is to understand everything first. This is commendable, but is likely to lead to lifelong learning and an incomplete degree. You have to accept that some aspects of your topic or research will remain unclear. Your adviser should always be able to tell which aspects you are expected to ignore for the time being. (You can always return to them in postgraduate work!) Therefore, be prepared to leave some topics untouched.

10.1.2 The other extreme: understanding the minimum.

You may think it is sufficient to walk a straight line from your current state of knowledge to your desired written Thesis, and hence save time by not going down blind alleys and by not taking any detours. There is no such straight line from here to there. You need to accept that hours will be spent on bits of the subject that will not be used in the Thesis, be it reading up on material or struggling with a new proof. Think of those alleys and detours as learning opportunities, rather than as wasted time—they ultimately lend you a broad view of the subject and contribute to your mathematical maturity.

10.1.3 How a Thesis is similar to an exam.

Like on an exam, you wish to present your knowledge in the best possible light, so make sure to organise your work around your stronger points and away from your weaker ones. Moreover, like in an exam, you should be able to defend every line in your Thesis a being "true" in the best mathematical sense.

10.1.4 How a Thesis is <u>not</u> similar to an exam.

In an exam you may be expected to provide an answer building on the first principles of the subject. In a Thesis you are expected to start at a much higher level and essentially choose your first principles (the results you can cite and use without proof). The balance of citations and proofs is also something you should discuss with your adviser.

10.1.5 Finding the balance: Understanding the purpose of your Thesis

In Section 7, we discussed the concept of the Ideal Reader and the notion of pitching your work to that Ideal Reader for best effect. Before you can think about the Ideal Reader, you have to understand, clearly and specifically, the purpose of your work—what it is you are aiming for. Here are some possible aims:

- Any old Thesis.
- An impressive Thesis.
- A well-written, concise Thesis that does some serious maths.
- A Thesis that shows you've put in the hours.
- A Thesis that displays your knowledge.
- A Thesis that will get you an excellent recommendation from your adviser.
- A Thesis that will get published.
- A Thesis that you can use to impress your parents/siblings/partner.
- A Thesis that you will be proud of.
- A Thesis that you will enjoy writing.
- A Thesis that you will enjoy burning after you are done.

All of these are acceptable goals[2], but they will not help you focus on writing a good Thesis. What you need to understand are the concrete requirements each type of thesis is meant to fulfil.

1. The aim of a BSc thesis is to show that the student has read up on a certain subject in sufficient depth and breadth, and is able to produce clear, well-structured exposition on the topic of their thesis that shows they understand the material.
2. The aim of an MSc thesis is to show that the student is capable of independent original research. This means displaying skills beyond those necessary for a BSc thesis by offering new insight into well-established problems (usually through proving existing theorems using different methods than the original proofs).
3. The aim of a PhD thesis it to show that the student has completed original research. This means displaying skills beyond those necessary for an MSc thesis by proving sufficiently many interesting new results.

Your Ideal Reader will be judging you against these aims, so your job is to write a Thesis which ticks all the expected items in a *conventional* way. The remainder of these Lecture Notes discuss what *conventional* means.

[2]What you think of your own Thesis and what you do with the hard copy in your free time is your own business.

10.2 The local structural elements

Each page of maths comprises some or all of the following elements:

i) named structural elements, such as Definitions or Theorems, that have strict, precise minimalist form;

ii) graphic elements, such as Tables or Figures;

iii) free-form exposition between these structural elements.

You can approach i) and ii) more or less independently, focusing on getting a proof right or drawing an appropriate graph. However, iii) requires you to have the other elements prepared, before you can proceed to bind them together and smooth the transition between them.

10.2.1 Named elements

You will already be familiar with reading named structural elements, but you may not have thought about their purpose and how you would distinguish between some of them. Here is the breakdown:

- *Definitions* are used for introducing important new notation and concepts. Less important notation or definitions that are being *recalled* rather than *introduced* are usually given in free-form exposition.

- *Theorems, Propositions, Lemmas*, and *Corollaries* signal noteworthy results. The names are indicative of their content.

  – Theorems either used to be or currently are the pinnacles of research; they are the weightiest and most important results on which further theory is built. Famous theorems get names such as the *Fundamental theorem of calculus, Fermat's Last Theorem,* the *Arzelá–Ascoli theorem*, etc.

  – Propositions are results that have lesser importance beyond the scope of the text. They do not become famous enough to be named.

  – Lemmas are usually technical statements used to prove more important results. Nonetheless, some lemmas do become famous enough to merit a name, such as *Dehn's lemma, Kronecker's lemma, Morse lemma.*

  – Corollaries are results of varying importance, but they must always follow fairly obviously from some previously stated result.

- *Proofs* normally follow immediately after the statement of the result, but can sometimes be delayed by explanatory exposition or, indeed, by Lemmas, Propositions, and other elements.

- *Remarks* signpost a variety of observations, side notes, caveats. Remarks are essentially bits of exposition that the writer wishes to highlight.

- The less commonly encountered *Conjectures* and *Questions* signpost speculation. Conjectures, like the *Riemann hypothesis*, are predictions of results left to others to prove. Questions are open-ended. You can either pose questions to others or your can cite other people's questions as motivation for your own work.

- *Footnotes* contain short asides that are unimportant for the main results of your work, but are still noteworthy in the context. Footnotes are rare in modern mathematics.

10.2.2 Exposition: Roadmap

A conventional element of any Introduction is what we call a *Roadmap*. It is a brief summary of the Thesis or paper that explains the main purpose and content of each section. Sometimes it is only a paragraph long; at other times, when the work is more complex, each section on its own may take up a whole paragraph. The shorter version is usually of the following form:

> In Section 2, we give a brief introduction to Knot Theory. In Section 3, we define the the Jones polynomial, discuss its properties, and compute a few examples. Finally, in Section 4, we use the Jones polynomial to prove a result about alternating links.

If you have not consciously noticed paragraphs like these before, from now on you certainly will. Every lecture of these Notes has a short introductory blurb that necessarily ends with a Roadmap.

The Roadmap is included regardless of whether your work has a Table of Contents. The Roadmap should not refer to or assume the reader is familiar with the Contents; instead, the Roadmap is meant to cap off the preceding text that introduced your work by laying out the steps through which you wish to accomplish your goal. The Contents list titles, while the Roadmap is an explanation of what is to come.

Finally, the Roadmap, with its formal section breakdown, conventionally appears only at the end of the Introduction. However, whenever a more complex program is being pursued within a section, local variations of the Roadmap are used to prepare the reader for the order in which things are done, or to indicate points of particular interest. You should not try to explain everything at every step, detailing definition, ideas, theorems before they occur in their natural order, but some hints may be welcome.

10.2.3 Exposition: Referencing other work

If you are not proving your own theorem, but are writing up someone else's in your own words, you should indicate the degree of modification.

- *This proof comes from Artin's book:* means that you virtually copied out the proof given in Artin's book.

- *This proof follows Artin's:* means that you are paraphrasing Artin's proof and preserving its structure.

- *This proof is modified from Artin's proof:* means that your proof is, in detail, significantly different from Artin's proof, but in structure comes from or follows that source with slight modification.

- *This proof is based on Artin's proof:* means that your proof follows the general outline of Artin's proof.

10.3 The global structural elements

A Thesis consists of a number of global structural elements. What follows is a list of these elements with brief commentary on their relative positions (if not as shown in the list), on their content, and on whether they are optional or not. The trickier elements we also discuss in separate sections below. Each entry in the list contains information about any relevant differences between the various theses and papers.

1. **Title Page.** Every thesis has one and it includes the university logo or header, the title of your Thesis, your name, the name of your adviser, and the date. Papers do not have Title Pages.

2. **Abstract.** The Abstract of the Thesis should appear either on the Title Page or on a separate page. It should not be more than a paragraph or two. In a paper, the Abstract appears directly underneath the title and name.

3. **Acknowledgements.** There are two kinds of acknowledgements: the personal (relatives, friends, pets) and the professional (adviser, institution, foundation). Personal acknowledgements do not appear in papers, but are quite common in PhD theses[3] and are then likely to appear on their own page after the Abstract. Professional acknowledgements are always welcome and should appear as the last element of the Introduction, or the last element before the Appendices; in BSc and MSc theses, the students are expected to write a single line thanking the adviser.

4. **Contents.** Shorter texts usually do not have a table of contents, but anything more than seventy or eighty pages will likely have one. In that case, you should let LaTeX generate this page automatically.

5. **Introduction.** This first chapter puts your work in a historical context or compares it to recent advances in research. It must contain a Roadmap. It usually ends with a small (professional) Acknowledgements section, as discussed.

6. **Notation.** For longer texts, or texts that use nonstandard notation, you would add a section specifying this notation, like a key on a map telling the reader which symbols means what. This section would go before or after the Introduction, or before or after the References.

7. **Preliminaries.** This second chapter should recall the basic material you will then build on.

[3]The authors of these Lecture Notes are of the opinion that personal acknowledgements should not appear in the other theses.

8. **The other chapters.**
9. **Conclusion or Outlook.** This would usually be a short discussion of future work or ideas. It is not that common to include such a section in a Thesis, and it would only be included in a paper if the authors had something they felt was significant to point out. This discussion would appear either at the end of the Introduction (before the Acknowledgements) or at the end of the last chapter.
10. **Appendix.** If your work calls for technical proofs or material that does not naturally fit in the flow of your Thesis, you may want to consider relegating it to one or more appendices. The appendices are usually labelled *Appendix A*, *Appendix B*, etc. A Thesis is unlikely to have more than a few appendices.
11. **References.** The list of all sources should be formatted to contain the name of the authors, the title, and the precise location of the source.

REMARK 10.4. It is common to double-space PhD theses, but check with your adviser whether this is the best practice at your university. It is not common to double-space MSc and BSc theses, as well as papers.

10.3.1 Title

At BSc and MSc thesis level, you are unlikely to have much choice regarding the title, and at PhD level the title will likely be a phrase related to your main result. You should agree the title with your adviser beforehand.

10.3.2 The Abstract

The Abstract complements the title, expanding on it and summarising the gist of your work. It should be a paragraph long, at most two, and rarely exceeds 200 words. Any new results should be mentioned.

Especially when it comes to papers, bear in mind that most people will only ever read the paper after they have read the Abstract and judged it to be sufficiently "interesting". To entice readers, you should phrase your results in an accessible, quick-to-read fashion; this means that you should avoid using technical jargon and introducing notation unless absolutely necessary.

Even though the Abstract appears first on the page, it is written last: after you have completed everything else.

Please avoid the following common mistakes when writing the Abstract:

- Not including the Abstract.
  (*This is bad form; every Thesis or paper must have an abstract.*)
- Including the Abstract, but not titling it, and leaving it as a standalone paragraph to float about on a page.
  (*This is bad form and confusing.*)
- Calling the Abstract anything other than *Abstract*, e.g. *An abstract*, *The Abstract*, *My Abstract*, *Preface*, *Summary*, *Introduction*, etc.
  (*This is unprofessional and potentially very confusing.*)

- Including the Roadmap in the Abstract.
(*The Roadmap belongs at the end of the Introduction.*)

- Including words of thanks in the Abstract.
(*Words of thanks belong in the Acknowledgements.*)

- Treating the Abstract as the Introduction.
(*The Abstract and the Introduction are two different structural elements that serve two different purposes. See Section 10.3.3 for information about the Introduction.*)

- Introducing symbols in the Abstract that are not used again in the Abstract.
(*This poses an unnecessary burden on the reader. It is redundant.*)

- Including long equations or specialised technical jargon.
(*Some jargon is unavoidable, but do not include your ϵ and δ proofs.*)

- Using expressions such as:
  - In this paper we would like to,
  - Down below we will attempt to,
  - Here I shall go on to show that, and so on.

  (*There is no need to refer to the paper or to the location of the information. The role of the Abstract is to describe what comes next, and therefore you may dispense with such references. Saying attempt in English, usually implies the attempt did not succeed, which is presumably rarely what you mean. As for wordiness, how to call yourself, and which tense to choose please refer to Section 7 for a discussion of these topics. In general, the Abstract is written in present simple tense.*)

10.3.3 The Introduction

The Introduction is there to introduce the problems you have tackled. It consists of two parts: the first is free-form; the second is structured to contain the Conclusion (optional), the Roadmap (mandatory), and the professional Acknowledgements (optional, though expected)—all of which we have discussed above.

The free-form part gives some historical background that leads up to the statement of the problem (or the other way around: you state the problem then explain its history). Then follows a discussion of a few notable attempt to solve the problem or a few notable (partial) solutions to the problem. Finally, there comes a summary of your approach, culminating in your own results. For a Thesis that has no new results, you can still go through all of these steps except for the last one; you simply end your Introduction after describing “your” approach to the problem, where *your* is most likely qualified as a modification or someone else’s work.

The Introduction should neither be sixty percent of the paper, nor one paragraph long. If you have fulfilled the goals outlined above, then it will likely be in the correct ballpark.

Like with the Abstract, the goal of the first few sentences of the Introduction is to entice the reader to keep going (when possible). If you are stuck for ideas for your first paragraph, take a look at Problem 2 on Problem Sheet below.

Here are a few first sentences that have been modified from real-life papers. They illustrate the *layout patterns* of information that you might expect in the Introduction of a professional paper.

- The history of Raindrop algorithms goes back at least forty years to the work of Cloud and Thunder, and their application to the Rainfall Problem.
  (*Aiming for a historical context; probably quite extensively.*)
- The first Tree-type equation was proposed by Yellow Seeds in 1953.
  (*Aiming for a briefer historical context.*)
- Labyrinth sets were introduced by A. House in [GH45].
  (*A similar, slightly more technical approach; no one would mistake this for a non-technical paper.*)
- Let D be the *Diamond Polygon of Matrices*, also defined as

  $$D := \{\diamond \triangleleft \Diamond \mid \diamond^2 = \Diamond\}.$$

  Then it is well known that D is not a Valuable Diamond.
  (*Jumps straight into defining the key concepts. This usually occurs with highly technical results that require many concepts to be defined before the problem can at all be stated.*

10.4 Writing and revision

There are many ways to write a Thesis or paper, but none of them involve starting with the Abstract or the Introduction—those you write last. One method involves the top-down approach, where you start with what you wish to prove (or have proved). You then look at what results you needed for your proof, then at what results those results used, and so on until you reach the first principles of your Thesis that you have determined beforehand based on a discussion with your adviser.

Outlining is important. Before you sit down to do some actual writing, you might think about outlining your work: decide on your preliminaries, decide on your notation, decide on a nearly complete list of all the other named elements, decide on any figures or tables you may need. Play around with the ordering in your outline until you find a flow that takes the reader from the first page to the end in the most logical, least choppy way. If you are happy with this order, then you can start filling out the various structural elements and leaving placeholder text for the free-form exposition.

Once a draft is complete, read it through, focusing mainly on the free-form exposition. The exposition is what gives your work a feeling of unity, and what makes a Thesis or paper more than just a list of results. Leave your work aside for a few weeks, then return to it with a fresh mindset—this should help you identify any problems either with the writing of with the maths. Finally, get a colleague in roughly the same field to look over your work, or at least, to look over the beginning. Ask them whether the Abstract and Introduction make sense (both should).

Everything we have discussed in these Lecture Notes should have helped prepare you for the task of successfully writing a good, coherent work in mathematics.

Good Luck!

Problem Sheet 10 (Non-examinable)

Go to https://arxiv.org, find your favourite subject area of maths, click on it, and open a dozen of the most recent uploads as PDF files.

Problem 1. **Getting a feel for Abstracts.**

Read the Abstracts of the twelve papers.

i) Pay particular attention to the first few words. Do you notice anything in common across the twelve papers?

ii) Look for how the following words are used: *we, show, prove, present, provide, result.*

iii) Note the average presence of notation, the length, and the tone of each Abstract.

iv) Read the Abstracts aloud, treating them as English sentences by inserting *blah* instead of any symbols.

It does not matter whether you understood what the Abstracts were saying mathematically, but reading them aloud encourages the ear to notice similarities in tone and sentence structure.

Problem 2. **Learning to write the beginning of an Introduction.**

Look at the Introductions of the twelve papers. Copy out (by hand or in LaTeX) the first few sentences of each Introduction.

i) Compare the different approaches the authors take.

ii) How would you describe these approaches? (For example, *historical context, context within the subject, diving into the details,* etc.) These descriptions should reflect the pattern used by the author to lay out the information.

iii) These layout patterns vary along a spectrum, but is there one that you find most appealing as a reader? Could you modify use it in your Introduction?

Your answers will depend on which subject you choose and what your personal preferences are. This Problem suggest that you to copy out the material because writing encourages you to think about the content from a perspective closer to that of an author. But be warned that we encourage you to use a layout pattern, not the actual words or sentences from other papers—that would be plagiarism!

Problem 3. Flick through the remainder of each of the papers, and note the layout of other global structural elements. Specifically, pay attention to finding the Roadmap and to how the authors transition between named structural elements.

Problem 4. Did you notice that any of the twelve papers violated the guidelines in these Lecture Notes? Excellent—this means you have understood what we have been saying! You should still follow our guidelines, however, unless there is a good reason why a particular alternative model offers a superior model.[4]

[4] If you are unsure, ask your adviser.